

CONCORDANCE OF KNOTS IN RATIONAL HOMOLOGY SPHERES

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This is an extended abstract of the author's recent results on concordance of knots in rational homology spheres. The purpose of this manuscript is to discuss main results and sketch some important ideas, without attempting to provide complete proofs. We will focus on results in higher dimensions, and discuss analogues in the classical dimension when the same results extend. Results peculiar to the classical dimension will be discussed in a separate paper. We work in the category of piecewise linear oriented manifolds.

1. Basic definitions. In this manuscript we study spherical n -knots (submanifolds isomorphic to S^n) in rational homology $(n+2)$ -spheres. Sometimes we will call such a knot K in Σ a *rational knot*, and denote it by a pair (Σ, K) . As in [2], we call two rational n -knots (Σ, K) and (Σ', K') are (*rationally*) concordant if there is a rational homology bordism W between Σ and Σ' and an embedded $S^n \times [0, 1]$ in W bounded by $K \cup -K'$. The set $\mathcal{C}_n^{\mathbb{Q}}$ of concordance classes of rational n -knots becomes an abelian group under the operation of connected sum of pairs. (Σ, K) represents the identity element of $\mathcal{C}_n^{\mathbb{Q}}$ if and only if there is a pair (Δ, D) of a rational ball Δ and an embedded disk D in Δ such that $\partial(\Delta, D) = (\Sigma, K)$. In this case we call K a *rational slice knot*.

The study of rational concordance has some remarkable motivations. First it is a natural generalization of classical notion of concordance of knots in the sphere, and so, we are interested in the difference and similarity between them. Furthermore, by Cochran and Orr [3], it was pointed out that there are numerous transfer maps from the set of concordance classes of codimension two links in the sphere to the rational knot concordance group $\mathcal{C}_n^{\mathbb{Q}}$. This application to link theory makes rational concordance of more interest. Finally rational concordance is closely related to the theory of rational homology surgery developed by Quinn [6] and Taylor and Williams [7]. (An unpublished note of Cochran and Orr discusses this relationship.) Investigation of rational concordance will be helpful in understanding the structure of rational homology surgery theory.

2. Known results. By Cochran and Orr, it was first discovered that there is a homomorphism of $\mathcal{C}_n^{\mathbb{Q}}$ into an algebraically defined group \mathcal{G}_n , which we will call the *algebraic rational concordance group*. We will define \mathcal{G}_n following the Seifert matrix approach of [2].

Theorem 1. *For any rational n -knot K with $n > 1$, there is a submanifold F in the ambient space bounded by c parallel copies of K for some positive integer c .*

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We call F a *generalized Seifert surface* of complexity c . In case of $n = 1$, we require that the framing on K induced by F must be equivalent to the framing used in taking parallel copies of K . Then there is a simple obstruction as follows.

Theorem 2. *For a rational 1-knot K , there exists a generalized Seifert surface if and only if the \mathbb{Q}/\mathbb{Z} -valued self-linking of K vanishes.*

For odd $n = 2q - 1 > 1$, we define the *Seifert pairing* $S: H_q(F; \mathbb{Q}) \times H_q(F; \mathbb{Q}) \rightarrow \mathbb{Q}$ in the usual way, using the \mathbb{Q} -valued linking number in the ambient space. For 1-knots admitting a generalized Seifert surface F , it turns out that there is an induced pairing on $\text{Coker}\{H_1(\partial F; \mathbb{Q}) \rightarrow H_1(F; \mathbb{Q})\}$, which we will call the Seifert pairing. In any case, choosing a basis, we obtain a *Seifert matrix* A .

As in [5, 4], we can define cobordism of rational Seifert matrices. For $c > 0$, let $G_{n,c}$ be the group of cobordism classes of Seifert matrices defined on generalized Seifert surface of complexity c for rational n -knots. Given a generalized Seifert surface F of complexity c , the union of r parallel copies of F can be viewed as a generalized Seifert surface of complexity rc . This gives rise to group homomorphisms $\phi_{c,rc}: G_{n,c} \rightarrow G_{n,rc}$ which form a direct system $(\{G_{n,c}\}, \{\phi_{c,rc}\})$. We define the *algebraic rational concordance group* \mathcal{G}_n to be the direct limit $\varinjlim G_{n,c}$ of this system.

Theorem 3. *For odd $n > 1$, there exists a homomorphism Φ_n of $\mathcal{C}_n^{\mathbb{Q}}$ into \mathcal{G}_n . For $n = 1$, Φ_1 is defined on the subgroup of classes of rational knots admitting generalized Seifert matrices.*

In [3], Φ_n was defined in terms of Blanchfield forms, and a signature invariant was defined using Φ_n . They used this to prove the long-standing conjecture that there are homology boundary links not concordant to boundary links. In [2] it was reformulated in terms of Seifert matrices. In [1], some applications to link theory were studied.

3. Invariants of \mathcal{G}_n . As our first main result, we discuss a complete set of invariants of \mathcal{G}_n . We call an algebraic integer z *reciprocal* if z and z^{-1} are conjugate over \mathbb{Q} , i.e., z and z^{-1} share the same irreducible polynomial. To define our invariants we consider the inverse limit P of reciprocal numbers; letting R_c be the set of reciprocal numbers for $c > 0$ and $R_{rc} \rightarrow R_c$ be the homomorphism $z \rightarrow z^r$, P is defined to be $\varprojlim R_c$. P can be identified with the set of sequences $\alpha = (\dots, \alpha_3, \alpha_2, \alpha_1)$ of algebraic numbers α_c such that $\alpha_{rc}^r = \alpha_c$ for any r and c . Let P_0 be the subset of P consisting of $\alpha = (\alpha_c)$ such that $|\alpha_c| = 1$.

Theorem 4. *For $\mathcal{A} \in \mathcal{G}_n$, there are invariants*

$$\begin{aligned} s(\mathcal{A}) &\in \mathbb{Z}^{P_0}, \\ e(\mathcal{A}) &\in (\mathbb{Z}/2)^P, \\ d(\mathcal{A}) &\in \varinjlim_c \prod_{\alpha \in P} \frac{\mathbb{Q}(\alpha^{1/c} + \alpha^{-1/c})^*}{\{z\bar{z} \mid z \in \mathbb{Q}(\alpha^{1/c})^*\}}, \end{aligned}$$

which determine \mathcal{A} completely, i.e., if $s(\mathcal{A}) = s(\mathcal{A}')$, $e(\mathcal{A}) = e(\mathcal{A}')$, and $d(\mathcal{A}) = d(\mathcal{A}')$ for $\mathcal{A}, \mathcal{A}' \in \mathcal{G}_n$, then $\mathcal{A} = \mathcal{A}'$.

Roughly speaking, our invariants of $\mathcal{G}_n = \varinjlim G_{n,c}$ are obtained by taking “limits” of known invariants of the matrix cobordism group $G_{n,c}$. In fact, our invariants s , e , and d can be viewed as limits of the signature, the modulo 2 residue class of the rank, and the discriminant of hermitian forms over $\mathbb{Q}(z)$, which is known to become invariants of $G_{n,c}$ when z is reciprocal. It turns out that these invariants have a contravariant naturality in the sense that the direct system defining \mathcal{G}_n is transformed into an inverse system of reciprocal numbers. This explains why the resulting invariants of \mathcal{G}_n are parametrized by the inverse limit P .

4. Computation of \mathcal{G}_n and the Artin Reciprocity. By computing and realizing our invariants of \mathcal{G}_n we show the following result:

Theorem 5. $\mathcal{G}_n \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty$.

This extends known results [3, 2] on the torsion-free part of \mathcal{G}_n , which were proved by using signature invariants. In computing the torsion part of \mathcal{G}_n , the invariants e and d play a crucial role. e is easier to compute since it lives in a simpler domain $(\mathbb{Z}/2)^P$, i.e., the value of e can be viewed as a sequence of elements in $\mathbb{Z}/2$ indexed by P . In case of d , its value lives in a more complicated domain

$$\varinjlim_c \prod_{\alpha \in P} \frac{\mathbb{Q}(\alpha^{1/c} + \alpha^{-1/c})^*}{\{u\bar{u} \mid u \in \mathbb{Q}(\alpha^{1/c})^*\}}.$$

This makes the computation of d much more sophisticated.

To compute d , we appeal to machinery of algebraic number theory. What follows is a brief outline of the idea. First we consider an easier problem: is a given element $x \in \mathbb{Q}(z + z^{-1})^*$ contained in $\{u\bar{u} \mid u \in \mathbb{Q}(z)^*\}$? From the observation that $\{u\bar{u} \mid u \in \mathbb{Q}(z)^*\}$ is equal to the set of norms of the field extension $\mathbb{Q}(z)/\mathbb{Q}(z + z^{-1})$, our problem is now how to detect norms. Appealing to the Hasse principle, this reduces to a local problem. In other words, x is a norm of $\mathbb{Q}(z)/\mathbb{Q}(z + z^{-1})$ if and only if x is a norm of $\mathbb{Q}(z)^v/\mathbb{Q}(z + z^{-1})_v$ for any valuation v of $\mathbb{Q}(z + z^{-1})$. This local problem can be investigated using the local Artin reciprocity, which is one of the central results in algebraic number theory. It asserts that there is an epimorphism

$$\varphi: \mathbb{Q}(z + z^{-1})_v^* \longrightarrow \text{Galois group of } \mathbb{Q}(z)^v/\mathbb{Q}(z + z^{-1})_v$$

whose kernel consists of norms of $\mathbb{Q}(z)^v/\mathbb{Q}(z + z^{-1})_v$. In order to compute our invariant d , which is a limit of discriminants of hermitian forms, we need to study the limiting behaviour of the local Artin maps φ .

While this establishes a framework for the computation of d , in general it seems a hard algebraic problem requiring deep investigation of a number theoretic phenomenon. Fortunately, in case of carefully constructed Seifert matrices, we can control the limiting behaviour of the Artin reciprocity. This enables us to prove the existence of infinitely many elements of order 2 and 4 in \mathcal{G}_n .

5. Structure of $\Phi_n: \mathcal{C}_n^{\mathbb{Q}} \rightarrow \mathcal{G}_n$. Regarding the question how much geometric information on $\mathcal{C}_n^{\mathbb{Q}}$ can be extracted via Φ_n , first we can observe that Φ_n is not injective. For example it does not distinguish the effect of the action of the rational homology cobordism group of rational spheres on $\mathcal{C}_n^{\mathbb{Q}}$ given by connected sum with ambient

spaces. To avoid such problems, we consider a subgroup of $\mathcal{C}_n^{\mathbb{Q}}$; let $b\mathcal{C}_n^{\mathbb{Q}}$ be the subgroup generated by knots in rational spheres bounding parallelizable rational balls. The following result shows that this subgroup plays a crucial role in understanding the structure of Φ_n .

Theorem 6.

- (1) For even n , $b\mathcal{C}_n^{\mathbb{Q}} = 0$
- (2) For odd $n > 3$, $\Phi_n|_{b\mathcal{C}_n^{\mathbb{Q}}}$ is an isomorphism onto \mathcal{G}_n .
- (3) For $n = 3$, $\Phi_n|_{b\mathcal{C}_n^{\mathbb{Q}}}$ is an isomorphism onto an index two subgroup of \mathcal{G}_n which is abstractly isomorphic to $\mathbb{Z}^{\infty} \oplus (\mathbb{Z}/2)^{\infty} \oplus (\mathbb{Z}/4)^{\infty}$.
- (4) For $n = 1$, $\Phi_n|_{b\mathcal{C}_n^{\mathbb{Q}}}$ is an epimorphism onto \mathcal{G}_n .

As a consequence, for odd $n > 1$,

$$0 \rightarrow \text{Ker } \Phi_n \rightarrow \mathcal{C}_n^{\mathbb{Q}} \rightarrow \text{Im } \Phi_n \rightarrow 0$$

splits and hence \mathcal{G}_n (or its index two subgroup if $n = 3$) is a summand of $\mathcal{C}_n^{\mathbb{Q}}$. In particular, $\mathcal{C}_n^{\mathbb{Q}}$ has elements of order 2, 4, and ∞ .

The triviality of $b\mathcal{C}_n^{\mathbb{Q}}$ for even n and the injectivity of $\Phi_n|_{b\mathcal{C}_n^{\mathbb{Q}}}$ for odd $n > 1$ are proved by using ambient surgery arguments. Given a generalized Seifert surface F of a rational n -knot in the boundary of a rational ball Δ , we do ambient surgery on F in Δ to simplify F . In contrast to the case of $\Delta = B^{n+3}$, we cannot kill homotopy groups of F even below the middle dimension, since Δ is not contractible. Instead, from the finiteness of homotopy groups, it turns out that we can kill torsion-free parts, so that a punctured rational homology disk is obtained (when the Seifert matrix obstruction vanishes for odd $n > 1$). By an additional argument using surgery along rational spheres, we can construct a desired slice disk in a rational ball.

The below result characterizes Seifert matrices of rational knots. It enables us to define \mathcal{G}_n purely algebraically in such a way that Φ_n is surjective.

Theorem 7. *A square matrix A over rationals can be realized by a Seifert matrix of a rational knot if and only if $P(A \pm A^T)P^T$ is integral and even unimodular for some nonsingular P . (In addition, for 3-knots, $16 \mid \text{sign}(A + A^T)$ is required.)*

6. Comparison with the ordinary knot concordance group. Let denote by $\mathcal{C}_n^{\mathbb{Z}}$ the ordinary concordance group of n -knots in S^{n+2} . By Levine's results [5, 4], $\mathcal{C}_n^{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}^{\infty} \oplus (\mathbb{Z}/2)^{\infty} \oplus (\mathbb{Z}/4)^{\infty}$ for odd $n > 1$. While the subgroup $b\mathcal{C}_n^{\mathbb{Q}}$ of $\mathcal{C}_n^{\mathbb{Q}}$ is abstractly isomorphic to the same group, the following result shows that it has very different structure from $\mathcal{C}_n^{\mathbb{Z}}$. For this purpose we consider the natural map $\mathcal{C}_n^{\mathbb{Z}} \rightarrow b\mathcal{C}_n^{\mathbb{Q}} \subset \mathcal{C}_n^{\mathbb{Q}}$. For even n , $\mathcal{C}_n^{\mathbb{Z}}$ is trivial and hence we have no interesting result. So we focus on odd n .

Theorem 8.

- (1) For odd n , $\text{Ker}\{\mathcal{C}_n^{\mathbb{Z}} \rightarrow b\mathcal{C}_n^{\mathbb{Q}}\}$ contains a subgroup isomorphic to $(\mathbb{Z}/2)^{\infty}$.
- (2) For odd $n > 1$, $\text{Coker}\{\mathcal{C}_n^{\mathbb{Z}} \rightarrow b\mathcal{C}_n^{\mathbb{Q}}\}$ contains a summand isomorphic to $\mathbb{Z}^{\infty} \oplus (\mathbb{Z}/2)^{\infty} \oplus (\mathbb{Z}/4)^{\infty}$.

To prove the first conclusion, we need to show that some order 2 elements of $\mathcal{C}_n^{\mathbb{Z}}$ are killed in $b\mathcal{C}_n^{\mathbb{Q}}$. For odd $n > 1$, we make use of purely algebraic arguments

appealing to $b\mathcal{C}_n^{\mathbb{Q}} \cong \mathcal{G}_n$ (or its index two subgroup). On the other hand, for $n = 1$, we use a Kirby calculus argument due to Cochran, which constructs directly the exterior of a slice disk in a rational 4-ball.

For the investigation of the cokernel, we heavily rely on a result that the invariants s, e, d of elements contained in $\text{Im}\{G_{n,c} \rightarrow \mathcal{G}_n\}$ have a special periodicity.

7. A Γ -group interpretation. Our results can be viewed from the viewpoint of homology surgery obstruction Γ -groups. In fact it can be seen that \mathcal{G}_n can be identified with

$$\Gamma_{n+3} \left(\begin{array}{ccc} \mathbb{Q}[\mathbb{Q}] & \xrightarrow{\text{id}} & \mathbb{Q}[\mathbb{Q}] \\ \text{id} \downarrow & & \downarrow \varepsilon \\ \mathbb{Q}[\mathbb{Q}] & \xrightarrow{\varepsilon} & \mathbb{Q} \end{array} \right).$$

From our results discussed above it follows that this Γ -group is isomorphic to $\mathbb{Z}^{\infty} \oplus (\mathbb{Z}/2)^{\infty} \oplus (\mathbb{Z}/4)^{\infty}$. Furthermore, as a consequence of our results on $\mathcal{C}_n^{\mathbb{Z}} \rightarrow b\mathcal{C}_n^{\mathbb{Q}} \subset \mathcal{C}_n^{\mathbb{Q}}$, the kernel and cokernel of the natural map

$$\Gamma_{n+3} \left(\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\text{id}} & \mathbb{Z}[\mathbb{Z}] \\ \text{id} \downarrow & & \downarrow \varepsilon \\ \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\varepsilon} & \mathbb{Z} \end{array} \right) \longrightarrow \Gamma_{n+3} \left(\begin{array}{ccc} \mathbb{Q}[\mathbb{Q}] & \xrightarrow{\text{id}} & \mathbb{Q}[\mathbb{Q}] \\ \text{id} \downarrow & & \downarrow \varepsilon \\ \mathbb{Q}[\mathbb{Q}] & \xrightarrow{\varepsilon} & \mathbb{Q} \end{array} \right)$$

contain a subgroup isomorphic to $(\mathbb{Z}/2)^{\infty}$ and a summand isomorphic to $\mathbb{Z}^{\infty} \oplus (\mathbb{Z}/2)^{\infty} \oplus (\mathbb{Z}/4)^{\infty}$, respectively.

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