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THE HOMFLY POLYNOMIAL OF KNOTS WITH FREE PERIODS

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1. Definitions and Examples

Definition 1. Let $p \geq 2$ be an integer. A link L in S^3 is said to be freely p-periodic if and only if there exists an orientation preserving diffeomorphism $h : S^3 \longrightarrow S^3$ such that:
1) h^i has no fixed points for all $1 \leq i \leq p-1$,
2) $h^p = Id_{S^3}$,
3) $h(L) = L$.

Example. The torus knot $5_1 = T(2,5)$ is the intersection of $\Sigma = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}; z_1^2 + z_2^5 = 0\}$ and S^3.

$$\begin{array}{rccc} h: & S^3 & \longrightarrow & S^3 \\ & (z_1, z_2) & \longmapsto & (e^{\frac{2i\pi}{3}} z_1, e^{\frac{2i\pi}{3}} z_2). \end{array}$$

$h(5_1) = 5_1$ hence 5_1 is freely 3-periodic.

Remark. Let $p \geq 2$ and q an integer. Consider the diffeomorphism $\varphi_{p,q}$ given by:

$$\begin{array}{rccc} \varphi_{p,q}: & S^3 & \longrightarrow & S^3 \\ & (z_1, z_2) & \longmapsto & (e^{\frac{2i\pi}{p}} z_1, e^{\frac{2iq\pi}{p}} z_2). \end{array}$$

If $\gcd(p,q) = 1$ then $\varphi_{p,q}$ is an orientation preserving diffeomorphism of order p and $\varphi_{p,q}^i$ has no fixed point for $1 \leq i \leq p-1$. Moreover, we have a p-fold cyclic covering $(\pi_{p,q}, S^3, L(p,q))$.

If $q = 0$ then $\varphi_{p,0}$ is a $2\pi/p$-rotation around a circle Δ. A link which is disjoint from Δ and invariant by $\varphi_{p,0}$ is said to be p-periodic.

Definition 3. *Let $p \geq 2$ and q an integer such that $gcd(p,q) = 1$. A link L of S^3 is said to be a (p,q)-lens link if and only if L is mapped onto itself by $\varphi_{p,q}$.*

Conjecture. *Let p be a prime and $h : S^3 \longrightarrow S^3$ an orientation preserving diffeomorphim of order p such that for all $1 \leq i \leq p-1$, h^i has no fixed points. Then there exists an integer q such that h is topologically conjugate to $\varphi_{p,q}$.*
This conjecture is solved for $p = 2$ and 3.

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2. Combinatorial description of freely periodic links

Let $n \geq 2$ and B_n the braid group.

$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \forall 1 \leq i \leq n-2 \rangle$.

Let $\Omega_n := (\sigma_1 \sigma_2 ... \sigma_{n-1})^n$ a generator of the center of the braid group $\mathcal{B}_n$.

Theorem 1. *A link K of S^3 is a (p,q)-lens link if and only if there exists an integer $n \neq 0$ and an n-tangle T such that*:

$$K = T^p (\sigma_1 \widehat{\sigma_2 ... \sigma}_{n-1})^{nq}.$$

In other words a (p,q)-lens link is obtained from a p-periodic link by q full twists along a disk.

Consequences.

1) The knot 9_{48} is a $(3,1)$-lens knot. Indeed $9_{48} = (\widehat{T\sigma_1^{-1}})^3 \sigma_1^2$ where T is the following 2-tangle:

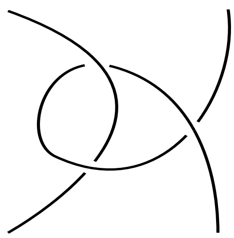

Figure 1

2) The torus link $T(n, nq + \alpha p)$ is a $(p,q)-$lens link. Furthermore, we can prove that:

$$T(n,m) \text{ is a } (p,q)-\text{lens link} \iff p \text{ divides } m - nq.$$

3. The HOMFLY polynomial

The HOMFLY polynomial is an invariant of ambiant isotopy of oriented links, defined by:

$$\begin{array}{ll} \textbf{(i)} & P_{\bigcirc}(v,z) = 1 \\ \textbf{(ii)} & v^{-1} P_{L_+}(v,z) - v P_{L_-}(v,z) = z P_{L_0}(v,z), \end{array}$$

where $\bigcirc$ is the trivial knot, L_+, L_- and L_0 are three oriented links which are identical except near one crossing where they look like in the following figure:

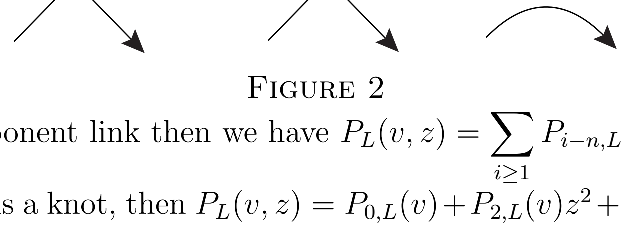

Figure 2

If L is an $n-$component link then we have $P_L(v,z) = \sum_{i \geq 1} P_{i-n,L}(v) z^{i-n}$.

In particular, if L is a knot, then $P_L(v,z) = P_{0,L}(v) + P_{2,L}(v) z^2 + \cdots + P_{2r,L} z^{2r}$ where

$P_{2i,L} \in \mathbf{Z}[v^{\pm 2}]$.

Notations. For p prime, let $\mathbb{F}_p = \mathbf{Z}/p\mathbf{Z}$, and $P_{2i,L}(v)_p$ the reduced modulo p of $P_{2i,L}(v)$.

Theorem 1. *Let p be an odd prime, $q = \pm 1$ and K a (p,q)-lens knot. Then $P_{0,K}(v)_p \in \Lambda_{p,q}$, where $\Lambda_{p,q}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{0,T(\alpha,\alpha q \pm p)}(v)_p$ for all $1 \leq \alpha \leq p-1$.*

Corollary 1. *Let K be a $(5,\pm 1)$-lens knot. Then $P_{0,K}(v)_5 = \sum a_{2i} v^{2i}$ where : $a_{10k+4} = 2a_{10k+2}$ and $a_{10k+6} = 2a_{10k+8}$ for all integer k.*

Theorem 2. *Let $p > 3$ be a prime, $q = \pm 1$ and K a (p,q)-lens knot. Then $P_{2,K}(v)_p \in \Gamma_{p,q}$, where $\Gamma_{p,q}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{2,T(\alpha,\alpha q \pm p)}(v)_p$ for all $1 \leq \alpha \leq p-1$.*

Corollary 2. *Let $q = \pm 1$ and K a $(5,q)$-lens knot. Then $P_{2,K}(v)_5 \in \mathbb{F}_5[v^{\pm 10}]$-module generated by v^{q8}.*

4. Sketch of the proof of Theorem 2.

Let T_+, T_- and T_0 be three $n-$tangles which are identical except near one crossing where they look like in the figure 2.

Notation: $D_+ := \widehat{T_+^p \Omega_n^q}$, $D_- := \widehat{T_-^p \Omega_n^q}$ and $D_0 := \widehat{T_0^p \Omega_n^q}$.

If D_+ is a knot, then D_- is also a knot. However there is two cases for D_0.
First case: D_0 has 2 components $D_1 \cup D_2$, each component is a (p,q)-lens knot.
Second case: D_0 has $(p+1)$-components $D_1 \cup D_2 \cup \cdots \cup D_{p+1}$, such that D_1 is a (p,q)-lens knot, the other components are identical and cyclically permuted by the lens action.

Lemma 1. *Let p be a prime. The following congruence holds modulo p:*

$$v^{-p} P_{D_+}(v,z) - v^p P_{D_-}(v,z) \equiv z^p P_{D_0}(v,z).$$

Lemma 2. *Let $p \geq 5$ be a prime.*
i) If D_0 has two components then:
$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = 0$
ii) If D_0 has $p+1$ components then:
$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = v^{2\lambda}(v^{-1} - v)^p P_{2,D_1}(v)(P_{0,D_2}(v))^p$.

Proof. From lemma 1,

$$v^{-p} P_{2,D_+}(v)_p - v^p P_{2,D_-}(v)_p = P_{3-(p+1),D_0}(v)_p.$$

$P_{3-(p+1),D_0}(v)$ can be computed by the Kanenobu-Miyazawa formula:

Let $n \geq 3$ be an integer and $L = l_1 \cup l_2 \cup \cdots \cup l_n$ an n-component link then:

$$P_{3-n,L}(v) = v^{2\lambda}(v^{-1}-v)^{n-2}\sum_{i<j}(v^{-2\lambda_{i,j}}P_{1,L_{i,j}}(v)\prod_{k\neq i,j}P_{0,l_k}(v)) -(n-2)v^{2\lambda}(v^{-1}-v)^{n-1}\sum_{i=1}^{n}(P_{2,l_i}(v)\prod_{j\neq i}P_{0,l_j}(v)),$$

where $L_{i,j}$ denotes the 2-component link $l_i \cup l_j$. $\lambda_{i,j} = lk(l_i, l_j)$ and λ is the total linking number of L.

Notation. Let $\Gamma'_{p,q}$ be the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{2,T(n,nq+p)}(v)_p$, for all n such that $gcd(n,p) = 1$.

The proof of theorem 2, will be done by induction on the number of crossings of the tangle T.

Lemma 3. $P_{2,D_+}(v)_p \in \Gamma'_{p,q}$ if and only if $P_{2,D_-}(v)_p \in \Gamma'_{p,q}$.

Lemma 4. *Every (p,q)-lens diagram may be transformed into a (p,q)-lens closed braid by a series of operations $D_+ \leftrightarrow D_-$ without increasing the number of crossings.*

Lemma 5. *Let B be an n-braid. The $(p,q)-$lens braid $B^p\Omega_n^q$ may be transformed into the torus knot $T(n, nq+p)$ by a series of operations $D_+ \leftrightarrow D_-$.*

Finally, we adapt the $D_+ \leftrightarrow D_-$ operation to torus knots of type $T(n, nq+p)$. By this operation we extract a finite set of generators for $\Gamma'_{p,q}$.

5. Applications

Let $p = 5$ and $q = 1$. We apply Theorem 1 and Theorem 2 to the 250 knots with less than 10 crossings.
The P_0-criterion does not decide for 48 knots.
The P_0-criterion does not decide for 6 knots, 4_1, 7_1, 8_{19}, 9_1, 10_{91}, 10_{104}.
If we combine both criteria, they fail to decide for three knots: 7_1, 8_{19} and 9_1.
But we know that: $7_1 = T(2, 2+5)$ and $9_1 = T(2, 4+5)$ are $(5,1)$-lens knots.
Also, $8_{19} = T(3,4)$, 5 does not divide 1=(4-3). Consequently 8_{19} is not a $(5,1)$-lens knot. However, it is a $(5,2)$-lens knot.

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