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ON BRAIDS, REPRESENTATIONS, AND HOMOTOPY THEORY

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Abstract. The purpose of this expository article is to describe some connections between braid groups, and homotopy theory, as well as features of related fibre bundles obtained from representations of pure braid groups.

1. Introduction

The main view of this article is that braids measure several related phenomenon through representations, homotopy groups, bundles, as well as cohomology groups with a smattering of number theory. This expository article describes some connections between these topics. The seven sections of this article are listed next.

1: Introduction
2: Braid groups, and classical homotopy groups
3: Lie algebras, and representations of the pure braid group
4: Representations of pure braid groups, and associated bundles
5: A sample computation, modular forms, and wild speculation
6: On long knots, braid groups, and recent work of R. Budney
7: Problems

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2. Braid groups, and classical homotopy groups

This section describes a close connection between braid groups, the homotopy groups of the 2-sphere, Vassiliev invariants, and the unstable Adams spectral sequence or Bousfield-Kan spectral sequence. The section is an exposition of how these different structures fit together in a natural way. The information here is based on joint work of J. Berrick, Y. L. Wong, J. Wu, and the author [12, 6, 28].

One way in which braid groups arise in homotopy theory is through the structure of a simplicial group. Recall that a simplicial group Γ_* is a collection of groups

$$\Gamma_0, \Gamma_1, \cdots, \Gamma_n, \cdots$$

together with face operations

$$d_i : \Gamma_n \to \Gamma_{n-1},$$

and degeneracy operations

$$s_i : \Gamma_n \to \Gamma_{n+1},$$

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for $0 \leq i \leq n$. These homomorphisms are required to satisfy the standard simplicial identities which themselves follow from features of the pure braid groups as given next.

An example of a simplicial group which in degree n is given by $\Gamma_n = P_{n+1}$, Artin's $(n+1)$-st pure braid group, is elucidated in [12]. The face operation d_i for $0 \leq i \leq n$ is given by deletion of the $(i+1)$-st strand, while the degeneracy operation s_j for $0 \leq j \leq n$ is gotten by "doubling" of the $(j+1)$-st strand. This last description of s_j is ambiguous, but explicit formulae for these homomorphisms are listed below as well as in [12]. The resulting simplicial group is denoted AP_*.

Recall that P_{n+1} is generated by symbols $A_{i,j}$ for $1 \leq i < j \leq n+1$ represented by a full twist of the j-th strand around the i-th strand. Artin's relations listed in [4, 20] are reformulated below in terms of commutators for which $[a,b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$.

(1) $[A_{r,s}, A_{i,k}] = 1$ for either $r < s < i < k$ or $i < k < r < s$,
(2) $[A_{k,s}, A_{i,k}] = [A_{i,s}^{-1}, A_{i,k}]$ for $i < k < s$,
(3) $[A_{r,k}, A_{i,k}] = [A_{i,k}^{-1}, A_{i,r}^{-1}]$ for $i < r < k$, and
(4) $[A_{r,s}, A_{i,k}] = [[A_{i,s}^{-1}, A_{i,r}^{-1}], A_{i,k}]$ for $i < r < k < s$.

The face operations in the simplicial group AP_* are defined as follows:

$$d_t(A_{i,j}) = \begin{cases} A_{i-1,j-1} & \text{if } t+1 < i, \\ 1 & \text{if t +1 = i}, \\ A_{i,j-1} & \text{if } i < t+1 < j \ , \\ 1 & \text{if t + 1 = j}, \\ A_{i,j} & \text{if } t+1 > i \ . \end{cases}$$

The degeneracy operations are defined as follows:

$$s_t(A_{i,j}) = \begin{cases} A_{i+1,j+1} & \text{if } t+1 < i \ , \\ A_{i,j+1} \cdot A_{i+1,j+1} & \text{if t+1 = i} \ , \\ A_{i,j+1} & \text{if } i < t+1 < j \ , \\ A_{i,j} \cdot A_{i,j+1} & \text{if t + 1 = j} \ , \\ A_{i,j} & \text{if } t+1 > j. \end{cases}$$

Simplicial groups admit structures analogous to those of topological spaces, but in a more rigid setting. For example, Moore [24] defined the homotopy groups of a simplicial group Γ_* by

$$\pi_n \Gamma_* = Z_n / d_0(C_{n+1})$$

where

$$Z_n = \cap_{0 \leq i \leq n} ker[d_i : \Gamma_n \to \ \Gamma_{n-1}],$$

and

$$C_{n+1} = \cap_{1 \leq i \leq n+1} ker[d_i : \Gamma_{n+1} \to \ \Gamma_n].$$

These homotopy groups are isomorphic to the classical homotopy groups of the geometric realization of the simplicial group Γ_*. In addition, these combinatorially defined groups agree with the classical homotopy groups of the realization of Γ_*.

There is a simplicial analogue of the topological construction given by the based loop space of a topological space. The loop-space of the simplicial group Γ_* denoted

$\Omega\Gamma_*$ is defined next as in [24] where $\Omega\Gamma_n$, the simplicial loop space in degree n, is the kernel of

$$d_0 : \Gamma_{n+1} \to \Gamma_n.$$

Define face and degeneracy operations $\bar{d}_i$, and $\bar{s}_i$ given by d_{i+1}, and s_{i+1} respectively for $i \geq 1$ by restriction to the subgroup $\Omega\Gamma_n$ in Γ_{n+1}. This formulation endows $\Omega\Gamma_*$ with the structure of a simplicial group. The groups $\pi_{n+1}\Omega\Gamma_*$, and $\pi_n\Gamma_*$ are naturally isomorphic as long as Γ_* is connected [24].

Recall Milnor's free group construction [23] given by $F[K_*]$ for a simplicial set K_* with a base-point $*$ in degree 0. Define $F[K_*]$ in degree n to be the free group generated by K_n modulo the single relation ${s_0}^n(*) = 1$. Thus the simplicial group $F[K_*]$ in degree n is isomorphic as a group to the free group generated by the set $K_n - \{{s_0}^n(*)\}$. Let $\Delta[1]$ denote the simplicial 1-simplex with S^1 the simplicial circle.

That the homotopy groups of AP_* are all trivial is easy to see by looking at examples of various natural braids. The next Threorem gives information about the loop space of AP_*.

Theorem 2.1. *The (simplicial) loop space of AP_* is isomorphic to $F[\Delta[1]]$ and is thus contractible. Hence $\pi_n AP_*$ is the trivial group for all n.*

A second result gives that the simplicial loop-space of S^2 is embedded naturally in AP_*. Recall that the simplicial circle S^1 in degree n is given by the simplicies $< 0^i, 1^{n+1-i} >$ for $0 \leq i \leq n$ with the relation that $< 0^{n+1} >=< 1^{n+1} >$ are equal to the base-point.

Theorem 2.2. *The exists a unique morphism of simplicial groups*

$$\Theta : F[S^1] \to AP_*$$

with $\Theta(< 0, 1 >) = A_{1,2}$. The map Θ is an embedding. Hence the homotopy groups of $F[S^1]$ are natural sub-quotients of AP_*, and the geometric realization of quotient simplicial set $AP_*/F[S^1]$ is homotopy equivalent to the 2-sphere. Furthermore, the smallest sub-simplicial group of AP_* which contains the element $\Theta(< 0, 1 >) = A_{1,2}$ is isomorphic to $F[S^1]$.

As an example, a braid which represents the classical Hopf map

$$\eta : S^3 \to S^2$$

in this framework is the braid given by the commutator $[x_1, x_2]$ for which

$$x_1 = A_{1,3} \cdot A_{2,3},$$

and

$$x_2 = A_{1,2} \cdot A_{1,3}.$$

The closure of the braid $[x_1, x_2]$ gives the Borromean rings. Next, consider the group of Brunnian braids $Brun_q$, those elements in P_q which are in the kernel of every projection map $P_q \to P_{q-1}$, that is those braids which become trivial after the deletion of any strand. The group $Brun_q$ surjects to $\pi_q S^2$ [12, 6]. Problems concerning the kernel of this last map are listed in section 7.

The method of proof of Theorem 2.2 is to show that the map $\Theta : F[S^1] \to AP_*$ is an embedding by a comparision of Lie algebras in which work of Toshitake Kohno

[17, 18], as well as Falk, and Randall [15] is essential. This classical method arises by replacing groups by Lie algebras as described in the next paragraph.

The q-th stage of the descending central series of a group π, $\Gamma^q = \Gamma^q(\pi)$ is defined to be the sub-group generated by all commutators of the form

$$[\cdots[x_1, x_2], x_3], \cdots], x_t]$$

for $t \geq q$. Then Γ^{q+1} is a normal sub-group of Γ^q with $E_0^q = E_0^q(\pi) = \Gamma^q/\Gamma^{q+1}(\Pi)$. There is a bilinear map

$$E_0^p \otimes E_0^q \rightarrow E_0^{p+q}$$

induced by the commutator function

$$[-,-] : \pi \times \pi \rightarrow \pi.$$

This pairing endows

$$E_0^*(\pi) = \oplus_{q\geq 1} E_0^q(\pi)$$

with the structure of a Lie algebra.

Consider the homomorphism $\Theta_n : F_n \rightarrow P_{n+1}$ on the level of associated graded Lie algebras obtained by filtering each of the groups by the descending central series. There is an induced morphism of Lie algebras

$$E_0^*(\Theta_n) : E_0^*(F_n) \rightarrow E_0^*(P_{n+1}).$$

The way in which these Lie algebras provide a tool for testing whether a group homomorphism is a monomorphism is described next.

Recall that a discrete group Γ is said to be residually nilpotent group if

$$\bigcap_{i\geq 1} \Gamma^i(\pi) = \{identity\}.$$

Proposition 2.3. (1) *Assume that π is a residually nilpotent group. Let*

$$\rho : \pi \rightarrow G$$

be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras

$$E_0^*(\rho) : E_0^*(\pi) \rightarrow E_0^*(G)$$

is a monomorphism. Then ρ is a monomorphism.

(2) *If π is a free group, and $E_0^*(\rho)$ is a monomorphism, then ρ is a monomorphism.*

An analysis of the morphism of Lie algebras obtained from the homomorphism $\Theta_n : F_n \rightarrow P_{n+1}$ with $\Theta_1(< 0, 1 >) = A_{1,2}$ then gives the result that Θ_n is a monomorphism [12]. There may be a more direct approach to this problem as the maps of Lie algebras here require some care. Two samples for which n is 2, or 3 are described next. In this case, the map Θ_n for $n = 2, 3$ are defined by setting

$$\Theta_2(< 0^i, 1^{3-i} >) = \begin{cases} A_{1,3} \cdot A_{2,3} & \text{if } i = 1, \\ A_{1,2} \cdot A_{1,3} & \text{if } i = 2, \text{ and} \end{cases}$$

$$\Theta_3(< 0^i, 1^{4-i} >) = \begin{cases} A_{1,4} \cdot A_{2,4} \cdot A_{3,4} & \text{if } i = 1, \\ A_{1,3} \cdot A_{2,3} \cdot A_{1,4} \cdot A_{2,4} & \text{if } i = 2, \\ A_{1,2} \cdot A_{1,3} \cdot A_{1,4} & \text{if } i = 3. \end{cases}$$

The map Θ_n is an embedding. Hence the homotopy groups of $F[S^1]$ are natural sub-quotients of AP_*, and the geometric realization of quotient simplicial set $AP_*/F[S^1]$ is homotopy equivalent to the 2-sphere. Furthermore, the smallest sub-simplicial group of AP_* which contains the element $\Theta(< 0, 1 >) = A_{1,2}$ is isomorphic to $F[S^1]$.

One other feature of the proof that $\Theta : F[S^1] \to AP_*$ is an embedding is that the filtration arising from descending central series is the method of Bousfield-Kan to construct the unstable Adams spectral sequence with the single modification obtained by replacing the descending central series by the mod-p descending central series. On the level of the Lie algebra obtained from these successive sub-quotients of the mod-p descending central series, this is precisely the E^0 term of the unstable Adams spectral sequence. On the other-hand, the Lie algebra given by $E_0^*(P_{n+1})$ above yields all of the Vassiliev invariants of pure braids via [17, 18].

Replacing pure braid group P_n by the pure braid group of the 2-sphere $P_n(S^2)$ results in a different, but similar construction denoted $AP_*(S^2)$. The projection maps give a structure analogous to that of a simplicial group which is a collection of groups with the face operations, but without the degeneracies. The resulting object is called a Δ-group.

In particular, a Δ-group has homotopy sets analogous to homotopy groups defined as the set of cosets above specified by

$$\pi_n AP_*(S^2) = Z_n/d_0(C_{n+1}).$$

In the case of the pure braid group for S^2, the following holds [6].

Theorem 2.4. *If $n \geq 4$, then $\pi_n AP_*(S^2)$ is a group which is isomorphic to $\pi_n(S^2)$.*

An analogue for all spheres arises at once by forming the coproduct of simplicial groups $AP_* \vee AP_*$ which in degree n is the free product $P_{n+1} \amalg P_{n+1}$. Then the smallest simplicial sub-group which contains $P_2 \amalg P_2$ in degree 1 has geometric realization given by $\Omega(S^2 \vee S^2)$. The space $\Omega(S^2 \vee S^2)$ has ΩS^n as a retract for any $n > 1$ by the Hilton-Milnor theorem.

Some problems related to the combinatorics of the above constructions are listed in section 7.

3. Lie algebras, and representations of the pure braid group

The Lie algebraic methods of the previous section also apply to the question of whether homomorphisms out of P_n are faithful. More precisely, the question of whether such representations are faithful reduce to a question at the level of Lie algebras. However, there is exactly one practical application of this process known to the author, and that is the proof of Theorem 2.2 here, a proof which is heavily dependent on the structure of various Lie algebras [12].

The group P_n is residually nilpotent. Thus a homomorphism $\Phi : P_n \to G$ is faithful if the induced map $E_0^*(\Phi) : E_0^*(P_n) \to E_0^*(G)$ is faithful. T. Kohno's analysis of $E^*_0(P_n)$ forces this Lie algebra to be "rigid" in a sense to be made precise by the next Theorem [11], joint work with S. Prassides, in which

- $\Delta = \Sigma_{1 \leq i < j \leq n} B_{i,j}$ for which $B_{i,j}$ is the projection of $A_{i,j}$ to $E^1_0(P_n)$, and
- $L[V_n]$ is the free Lie algebra generated by $\{B_{1,n}, B_{2,n}, \cdots, B_{n-1,n}\}$ the image of the Lie algebra determined by those pure braids which are trivial after deleting the last strand.

Theorem 3.1. *Let*

$$\Phi : P_n \to G$$

be a homomorphism.

If the maps of Lie algebras

$$E^*_0(\Phi)|_{L[V_n]} : L[V_n] \to E^*_0(G),$$

and

$$E^*_0(\Phi)|_{L[\Delta]} : L[\Delta] \to E^*_0(G)$$

are both monomorphisms, then Φ is a monomorphism.

The adjoint representation

$$Ad : E^*_0(P_n) \to Der^{Lie}_*(E^*_0(P_n))$$

defined by

$$Ad(X)(L) = [X, L]$$

restricted to an element L in $L[B_{1,n}, B_{2,n}, \cdots, B_{n-1,n}]$ gives a linear map

$$Ad : E^*_0(P_n) \to L[B_{1,n}, B_{2,n}, \cdots, B_{n-1,n}].$$

A direct corollary of the proof of the previous Theorem is stated next.

Theorem 3.2. *The kernel of adjoint representation*

$$Ad : E^*_0(P_n) \to Der^{Lie}_*(L[V_n])$$

is $L[\Delta]$.

It remains to be seen whether these Lie algebraic constructions can be used to inform on representations of residually nilpotent groups.

4. Representations of pure braid groups, and associated bundles

Given a discrete group Γ together with a topological group G, consider the space of all homomorphisms

$$Hom(\Gamma, G).$$

The quotient space modulo the conjugation action by inner automorphisms

$$Rep(\Gamma, G)$$

has some similar features, but will not be addressed here. In case Γ is the pure braid group, the topology of these spaces have natural properties discussed in this section which is an account of joint work of A. Adem, D. Cohen, and the author [1, 2].

Recall that the classifying space of a group G, written BG, is the quotient of a contractible space EG by a free, properly discontinuous action of G. If G is discrete, then $BG = K(G, 1)$.

The classical example of G given by the orthogonal group $O(n)$ is described next for purposes of exposition. Let $V(n,k)$ denote the Stiefel manifold of n-frames in $\mathbb{R}^{n+k}$ with $V(n,k)$ regarded as a subspace of $V(n,k+1)$. Let $Gr(n,k)$ denote the Grassmann manifold of n-planes through the origin in $\mathbb{R}^{n+k}$ with $Gr(n,k)$ regarded as a subspace of $Gr(n,k+1)$. Then $EO(n) = \cup_{k\geq 0} V(n,k)$,$BO(n) = \cup_{k\geq 0} Gr(n,k)$, and $BO(n) = EO(n)/O(n)$.

Consider the universal n-plane bundle

$$EO(n) \times_{O(n)} \mathbb{R}^n \rightarrow \ BO(n).$$

If G is any subgroup of the classical orthogonal group $O(n)$, then the pull-back of the universal bundle is that bundle given by

$$EO(n) \times_G \mathbb{R}^n \rightarrow \ EO(n)/G$$

with fibre $\mathbb{R}^n$ where $BG = EO(n)/G$.

In addition, given a representation $\rho : \Gamma \rightarrow \ G$, there is a bundle obtained over $B\Gamma$ obtained pulling back the universal bundle. That bundle is $E\Gamma \times_\Gamma \mathbb{R}^n \rightarrow \ B\Gamma$ in which Γ acts on $\mathbb{R}^n$ via the representation ρ. In addition, there are natural evaluation maps

$$e : Hom(\Gamma, G) \rightarrow \ [B\Gamma, BG]$$

for which $[B\Gamma, BG]$ denotes the set of pointed homotopy classes of maps. The evaluation map e is defined by sending an element f to $e(f)$ the homotopy class of the induced map on the level of classifying spaces.

In case Γ is a discrete group, the classifying space $B\Gamma$ is a $K(\Gamma, 1)$, and there are two natural questions which arise naturally from work of T. Kohno on Vassiliev invariants for which Γ is the pure braid group [17, 18].

(1) For which representations $\rho : \Gamma \rightarrow \ G$ are the resulting bundles isomorphic to the trivial bundle ?
(2) What is the topology of the space $Hom(\Gamma, G)$, and $Rep(\Gamma, G)$ when G is one of the classical Lie groups such as $O(n)$, $GL(n,\mathbb{R})$, or $PGL(n,\mathbb{R})$, or $Sp(2n,\mathbb{R})$?

The first crude question above, whether the bundle itself is a product as a bundle, admits further questions associated to properties satisfied by flat connections [17, 18]. It may be interesting to see what additional information is encoded in representations which give rise to trivial bundles.

is the configuration space of ordered k-tuples of distinct points in the complex line, $Conf(\mathbb{R}^2, k)$. In the case of complex hyperplane complements, the initial question as to whether the associated bundles are trivial admits the following answer [1].

This theorem admits a description in a more classical context.

In subsequent extensions of this result, the following feature arises [2]. If Γ is the pure braid group, or more generally the fundamental group of the complement of a complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^k$, then the question of whether a bundle is trivial is closely tied to the structure of the maximal abelian subgroups of a Lie group G, those abelian subgroups which are not proper subgroups of any other abelian subgroup of G. One subsequent result is as follows [2].

Theorem 4.1. *Assume that*

(1) *$\rho : \Gamma \to G$ is any representation of the fundamental group of the complement of a complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^k$, and*
(2) *G is any topological group with the property that every maximal abelian subgroup is path-connected.*

Then the element $B\rho : B\Gamma \to BG$ regarded as an element of $[B\Gamma, BG]$ is the trivial element.

If G is a compact Lie group with a connected maximal torus, then the hypotheses of the theorem are satisfied. Examples are $SU(n)$, $U(n)$, but not $SO(n)$, or $O(n)$.

A related question is to ask about the subgroup of the real K-theory of $B\Gamma$ which these bundles generate. This subgroup denoted $KO^0_{rep}(B\Gamma)$ is addressed in [1].

Theorem 4.2. *Let Γ be the fundamental group of the complement of a $K(\Gamma, 1)$ arrangement and let ζ_1 and ζ_2 be arbitrary classes in $H^1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ and $H^2(\Gamma; \mathbb{Z}/2\mathbb{Z})$. Then there is a finite dimensional orthogonal representation of Γ with first and second Stiefel-Whitney classes given by ζ_1 and ζ_2 respectively. Moreover for these groups Γ, the Stiefel-Whitney classes induce an isomorphism of groups*

$$KO^0_{rep}(B\Gamma) \cong H^1(\Gamma, \mathbb{Z}/2) \oplus H^2(\Gamma, \mathbb{Z}/2).$$

Notice that the number of elements in $H^1(\Gamma, \mathbb{Z}/2) \oplus H^2(\Gamma, \mathbb{Z}/2)$ gives a lower bound for the number of path components for $Hom(\Gamma, G)$. It is then natural to ask for conditions which imply that the map

$$e : Hom(\Gamma, G) \to [B\Gamma, BG]$$

induces an isomorphism on the level of sets

$$E : \pi_0 Hom(\Gamma, G) \to [B\Gamma, BG].$$

There are two notable examples where this last map is an isomorphism:

(1) Work of [19] implies that if Γ denotes the fundamental group of a closed, compact, orientable Riemann surface of genus at least 2, and G denotes a connected, compact, semi-simple Lie group, then E is an isomorphism of sets.
(2) Deep work of Miller, and Lannes gives that E is an isomorphism in case Γ is an elementary abelian p-group, and G is a compact Lie group.

In addition, $\pi_0 Hom(\Gamma, G)$ is a single point when Γ is a finitely generated free abelian group of rank $k > 0$, and G is $U(n)$. However in case n is much larger than k with $k > 1$, $[B\Gamma, BG]$ is of infinite cardinality. Thus E fails to be an isomorphism of sets in these cases.

The last part of this section addresses one related example given by certain choices of mapping class groups Γ with $G = SO(3)$. Let $Conf(S^2, k)$ denote the configuration space of ordered k-tuples of distinct points in the 2-sphere S^2. The symmetric group on k letters Σ_k acts naturally by permutation of coordinates. Write $Conf(S^2, k)/\Sigma_k$ for the quotient.

Then $SO(3)$, and $PGL(2, \mathbb{C})$ act naturally on the space of complex lines through the origin in complex 2-space $S^2 = \mathbb{CP}^1$. Hence $SO(3)$ as well as $PGL(2, \mathbb{C})$ act diagonally on $Conf(S^2, k)/\Sigma_k$. Form the Borel construction

$$ESO(3) \times_{SO(3)} Conf(S^2, k)/\Sigma_k$$

together with the natural projection map

$$p_k : ESO(3) \times_{SO(3)} Conf(S^2, k)/\Sigma_k \to\ ESO(3)/SO(3) = BSO(3).$$

It was proven in [10] that $ESO(3) \times_{SO(3)} Conf(S^2, k)/\Sigma_k$ is $K(\Gamma_0^k, 1)$ if $k \geq 3$ where Γ_0^k is the mapping class group for genus zero surfaces with k punctures. Thus the projection map above is a map

$$p_k : B\Gamma_0^k \to\ BSO(3)$$

for $k \geq 3$.

If $k \geq 2$, then the map $p_{2k} : B\Gamma_0^{2k} \to\ BSO(3)$ satisfies the condition that the mod-2 cohomology of $BSO(3)$ injects in that for $B\Gamma_0^{2k}$, and so w_1, and w_2 are both non-zero [10]. However, the map p_k is not homotopic to $B\rho$ for any representation $\rho : \Gamma_0^k \to\ SO(3)$ in case $k \geq 6$ [5]. It is natural to ask whether the composite

$$B\Gamma_0^6 \to\ BSO(3) \to\ BPGL(2, \mathbb{C})$$

induced by the inclusion of the maximal compact subgroup $SO(3) \to\ PGL(2, \mathbb{C})$ is homotopic to a map $B\rho'$ induced by a representation $\rho' : \Gamma_0^6 \to\ PGL(2, \mathbb{C})$.

As a parenthetical remark, the fibre bundle $B\Gamma_0^n \to\ BSO(3)$ admits an interpretation as a classical "incidence bundle" assuming $n > 2$. That is, consider $\tilde{G}r(3, k)$ the Grassmann manifold of oriented 3-planes through the origin in $\mathbb{R}^{k+3}$ together with the space of ordered pairs

$$A(3, k, n) = \{(V; (x_1, \cdots, x_n))| V \epsilon \tilde{G}r(3, k), (x_1, \cdots, x_n) \epsilon Conf(S(V), k)/\Sigma_k\}$$

where $S(V)$ denotes the unit sphere in V. Then $A(3, k, n)$ is "naturally" a subspace of $A(3, k + 1, n)$. The union

$$A(3, \infty, n) = \cup_{k \geq 0} A(3, k, n)$$

is a $K(\pi, 1)$ with $\pi = \Gamma_0^n$ for $n > 2$ [5].

In addition, there is an analogous fibre bundle for the mapping class group Γ_2 for genus 2 surfaces. That bundle is given by

$$Conf(S^2, 6) \times_{\Sigma_6} S^1 \to\ B\Gamma_2 \to\ BSU(2)$$

where Σ_6 acts on S^1 by the sign of a permutation [10]. It is natural to ask whether this last fibre bundle is obtained from the natural symplectic representation $\Gamma_2 \to Sp(4, \mathbb{R})$.

5. A sample computation, modular forms, and wild speculation

Consider the mapping class group Γ_g for a closed orientable surface S_g of genus g together with the symplectic representation obtained by evaluating a diffeomorphism on the first homology group of the surface

$$\Phi_g : \Gamma_g \to\ Sp(2g, \mathbb{Z}).$$

This section gives a connection between the maps

$$\Theta_n : F_n \to\ P_{n+1}$$

of section 2, the maps

$$\Phi : B_{2g+2} \to\ Sp(2g, \mathbb{Z})$$

obtained from the classical symplectic representation of the mapping class groups for surfaces of genus g together with restrictions to the image of braid groups in Γ_g, and their connection to classical number theoretic constructions. This section is based on joint work with J. Wu [12].

There are maps $B_{2g+2} \to \Gamma_g$ obtained from the centralizer of the hyperelliptic involution which are constructed by Dehn twists along a "necklace" of circles along S_g analogous to the constructions given on pages 183-188 in [7]. Restrict to $g = 1$ to obtain

$$B_3 \to B_4 \to SL(2, \mathbb{Z}).$$

Next, consider the map obtained from the maps $\Theta_n : F_n \to P_{n+1}$ described in section 2. In the special case of $n = 2$, there is a composite of $\Theta_2 : F_2 \to P_3$ given by

$$\Lambda_2 : F_2 \to PSL(2, \mathbb{Z})$$

which is induced by the natural composite of Θ_2 with the symplectic representation $B_3 \to SL(2, \mathbb{Z})$.

The image of F_2 is described in the next result for which the principal congruence subgroup of level q in $PSL(2, \mathbb{Z})$, written here as $\Gamma(q)$, is the kernel of the mod-q reduction map $PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/q\mathbb{Z})$.

Theorem 5.1. *The homomorphism $\Lambda_2 : F_2 \to PSL(2, \mathbb{Z})$ maps F_2 isomorphically onto the principal congruence subgroup of level 2 in $PSL(2, \mathbb{Z})$, $\Gamma(2)$.*

A proof of this theorem is given at the end of this section for completeness. A discussion of features of this map for higher genus surfaces is given next. The analogous computation gives that the image of the composite

$$F_{2g+1} \xrightarrow{\Theta_{2g+1}} P_{2g+2} \xrightarrow{inclusion} B_{2g+2} \xrightarrow{\Phi} Sp(2g, \mathbb{Z}) \xrightarrow{reduction} PSp(2g, \mathbb{Z})$$

lies in the principal congruence subgroup given by the kernel of the mod-2 reduction map

$$PSp(2g, \mathbb{Z}) \longrightarrow PSp(2g, \mathbb{Z}/2\mathbb{Z}).$$

The main point here is whether it there is natural structure here which distinguishes braids representing elements in homotopy. In particular, homomorphisms with source given by the pure braid group, and with target an abelian group do not provide enough information to distinguish such braids. On the other-hand, crossed homomorphisms from the braid group to abelian groups provide more information as well as interpretations in terms of classical number theory.

Recall that if an abelian group V is a left G-module for a discrete group G, then a crossed homomorphism is (1) a function (not necessarily a homomorphism)

$$d : G \to V$$

for the G-module V which satisfies (2)

$$d(gh) = d(g) + gd(h).$$

Classical group cohomology gives that $H^1(G; V)$ is isomorphic to the quotient group of the crossed homomorphisms modulo the principal crossed homomorphisms.

One classical example arises in this context, and in fact distinguishes certain braids representing homotopy elements. Namely, let V_2 denote the "tautological" $SL(2,\mathbb{Z})$-module given by

$$V_2 = (\oplus_2 \mathbb{Z}) \otimes \mathbb{R}$$

with the action of $SL(2,\mathbb{Z})$ specified by left translation by matrices. Next, consider the k-fold symmetric power of V_2, $Sym^k(V_2)$, the space of homogeneous polynomials of degree k in two indeterminates x_1, and x_2. Thus $Sym^k(V_2)$ is naturally an $SL(2,\mathbb{Z})$-module.

The Eichler-Shimura isomorphism [14, 27] specifies an $\mathbb{R}$-linear isomorphism of real vector spaces given by

$$H^1(SL(2,\mathbb{Z}); Sym^{2k}(V_2^{\mathbb{R}})) \to \ M_{2k+2}$$

for $k \geq 0$, and

$$H^1(SL(2,\mathbb{Z}); Sym^{2k+1}(V_2^{\mathbb{R}})) = \{0\}$$

for which M_{2k+2} denotes the real vector space of modular forms of weight $2k+2$ based on the standard $SL(2,\mathbb{Z})$ action on the upper 1/2-plane by linear fractional transformations. This identification was used in work of Furusawa, Tezuka, and Yagita [16] to determine the real cohomology of $BDiff^+(S^\times S^1)$.

Analogous computations apply to both B_3, and F_2. The Eichler-Shimura isomorphism identifies certain crossed homomorphisms out of $SL(2,\mathbb{Z})$, and thus both B_3, as well as F_2. These results provide modular forms which regarded as crossed homomorphisms take different values for braids representing different elements in $\pi_3 S^2$. This type of computation, an easy exercise in classical homotopy, raises the question of whether analogous modular forms for higher genus surfaces distinguish braids, or elements in the free group representing homotopy classes via Theorems 2.2, and 2.4.

It is natural to ask whether the cycles in $F[S^1]$ regarded as a subgroup of AP_* are "distinguished" by natural choices of crossed homomorphisms for some choice of module V. Given that the homotopy groups of spheres are natural subquotients of braid groups and their free products as described in section 2, it is natural to try to measure these features of braids. This last paragraph is the "wild speculation" as given in the title of this section.

To close this section, a proof of Theorem 5.1 is given next.

Proof. This proof, a direct calculation, is listed next. The braid group B_3 is generated by two elements σ_i for $i = 1, 2$ for which the following are satisfied.

$$\Phi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$\Phi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Furthermore, the formulas below hold in B_3.

(1) $A_{1,2} = \sigma_1^2$,
(2) $A_{2,3} = \sigma_2^2$, and
(3) $A_{1,3} = \sigma_2 \cdot \sigma_1^2 \cdot \sigma_2^{-1}$.

The image of these elements in $SL(2,\mathbb{Z})$ is determined by

$$\Phi(A_{1,2}) = \Phi(\sigma_1^2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$\Phi(A_{2,3}) = \Phi(\sigma_2^2) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

$$\Phi(A_{1,3}) = \Phi(\sigma_2 \cdot \sigma_1^2 \cdot \sigma_2^{-1}) = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix},$$

and

$$\Phi(\sigma_2^{-1}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Recall the map $\Theta_2 : F_2 \to P_3$ which satisfies

(1) $\Theta_2(x_1) = A_{1,3} \cdot A_{2,3}$, and
(2) $\Theta_2(x_2) = A_{1,2} \cdot A_{1,3}$.

Thus the image of F_2 in $SL(2,\mathbb{Z})$ is specified by

$$\Lambda_2(x_1) = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix},$$

and

$$\Lambda_2(x_2) = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}.$$

Notice that the values of $\Lambda_2(x_i)$ for $i = 1, 2$ give a basis for the free group on 2 generators isomorphic to $\Gamma(2)$ □

6. On long knots, braid groups, and recent work of R. Budney

The results stated in this section are mildly peripheral to the results in earlier sections, but are closely connected with the braid groups, and the mathematics at this conference.

Let $\mathcal{K}_3$ denote the space of long knots in $\mathbb{R}^3$. That is, $\mathcal{K}_3$ is the space of smooth embeddings $f : \mathbb{R}^1 \to \mathbb{R}^3$ which go off to $\pm\infty$ in a standard way outside of the closed interval $[-1, 1]$, and which when restricted to $[-1, 1]$ have image contained in the unit ball. Certain fruitful constructions yielding double loop spaces are closely connected to $\mathcal{K}_3$. These constructions are addressed next.

Consider the configuration space $Conf(\mathbb{R}^2, q)$ with the natural action of the symmetric group on q letters Σ_q. Given any space X, the symmetric group Σ_q acts on X^q by permutation of coordinates. Form the homotopy orbit space also known as the Borel construction

$$Conf(\mathbb{R}^2, q) \times_{\Sigma_q} (X)^q.$$

The fundamental group of $Conf(\mathbb{R}^2, q)/\Sigma_q$ is denoted B_q Artin's braid group with q strands. In case $X = K(\pi, 1)$, then $Conf(\mathbb{R}^2, q) \times_{\Sigma_q} (X)^q$ is a $K(G, 1)$ for which the fundamental group is the analogue of the classical wreath product construction given by the following split group extension

$$1 \to \pi^q \to G \to B_q \to 1.$$

This last construction, used extensively in homotopy theory, is central to [13, 21, 22, 26, 9]. Let $X \amalg Y$ denote the disjoint union of topological spaces X, and Y. Then the disjoint union

$$\amalg_{q \geq 0} Conf(\mathbb{R}^2, q) \times_{\Sigma_q} (X)^q$$

is a "good" approximation for certain double loop spaces as explained next.

Write $C_2(X \amalg \{*\}) = \amalg_{q \geq 0} Conf(\mathbb{R}^2, q) \times_{\Sigma_q} (X)^q$. If the space X is of the homotopy type of a CW-complex, there is a map

$$\Phi : C_2(X \amalg \{*\}) \to \ \Omega^2\Sigma^2(X \amalg \{*\})$$

which is an embedding in homology where $X \amalg \{*\}$ denotes the disjoint union of the space X, and a point $*$. By inverting the action of $\pi_0(C_2(X \amalg \{*\})) = \pi_0$ on the homology of $C_2(X \amalg \{*\})$, there is an induced isomorphism

$$H_*(C_2(X \amalg \{*\}); \mathbb{F})[1/\pi_0] \to \ H_*(\Omega^2\Sigma^2(X \amalg \{*\}); \mathbb{F})$$

for any field coefficients $\mathbb{F}$ [26, 9]. In addition, there are further homotopy equivlances which are not discussed here [21].

Building on work of A. Hatcher, R. Budney establishes a connection between braid groups, long knots as well as the constructions $C_2(X \amalg \{*\})$ which approximate the double loop space of a double suspension in [8] in Theorem 3.20. Let P denote the isotopy class of a prime long knot, and let X_P denote the path component in $\mathcal{K}_3$ of the class of the prime long knot P. Finally, let $X_{\mathcal{K}}$ denote the disjoint union

$$\amalg_P X_P$$

where P runs over all isotopy classes of prime knots P (excluding the trivial long knot).

Theorem 6.1. *There exists a homotopy equivalence*

$$\Phi : C_2(X_{\mathcal{K}} \amalg \{*\}) \to \ \mathcal{K}_3.$$

Budney's Theorem addresses the structure of the space of long knots as well as analogues of spaces of long knots. Using work of Jaco, Shalen, Johannson, and Thurston, Budney gives more detailed information concerning the structure of the isotopy classes of prime knots P. These are generated by operations on the trivial knot by

(1) cabling,
(2) connected sums, and
(3) hyperbolic satellite operations.

The spaces X_P are all $K(\pi, 1)$'s for which Budney gives some explicit identifications of π. His methods also inform on other analogous spaces of smooth embeddings [8]. It seems likely that the above results should provide methods for analyzing the homology of $\mathcal{K}_3$, as well as related spaces of long knots.

7. Problems

Problem 1: Consider the simplicial group $F[S^1]$ together with the relationship of the braid group, and the homotopy groups of the 2-sphere as given in section 2 here. It is known that if $n > 3$, then $\pi_n S^2$ is a finite abelian group. Furthermore, there are no elements of order 8 or p^2 if p is an odd prime.

Does this bound on the order of torsion correspond to natural structures associated to pure braids, or to Brunnian braids ?

Does the known structure of the homotopy groups of the 2-sphere impact the structure of braids ?

Problem 2: Consider cosets of elements in F_n which are cycles. Each such coset specifies exactly one element in homotopy. Do the braids in a fixed coset admit global distinguishing geometric properties ?

Problem 3: The cycles in P_{n+1} are precisely the Brunnian braids. Is there a natural geometric meaning for braids which are a boundary in F_n ?

Problem 4: What is the kernel of the Gassner representation when restricted to the subgroup generated by the elements $\{A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}\}$?

Problem 5: What is the geometry of $Hom(\Gamma, G)$ in case $\Gamma = P_n$, and G is a Lie group such as $O(n)$ or $SL(n, \mathbb{R})$? A lower bound for the number of components for $Hom(P_n, O(q))$ for q large compared to n is the number of elements in $H^1(P_n, \mathbb{Z}/2) \oplus H^2(P_n, \mathbb{Z}/2)$. Is this lower bound sharp ?

Problem 6: What is the homology of the space of long knots ?

Problem 7: Consider homomorphisms $\gamma_g : B_{2g+2} \rightarrow \Gamma_g$ obtained via Dehn twists for a "necklace" of circles along the surface as described in [7, 10] or in [5], page 1.

Restrict γ_g to the group π given either by P_{2g}, F_{2g-1}, P_{2g-1}, or F_{2g-2} to obtain a representation

$$\pi \rightarrow B_{2g+2} \rightarrow Sp(2g, \mathbb{Z}).$$

What is the real cohomology of π with coefficients in the symmetric powers of tautological symplectic representation $V_{2g} = \oplus_{2g}(\mathbb{Z}) \otimes \mathbb{R}$? Do these classes distinguish natural families of braids ? Do these classes admit an interpretation in terms of modular forms ? Identify whether, and how elements in π are "distinguished" by crossed homomorphisms representing elements in $H^1(\pi; Sym^k(V_{2g}))$.

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