Proceedings of the East Asian School
of Knots, Links, and Related Topics
February 16–20, 2004 • Seoul, Korea

ON DIFFEOMORPHISMS OVER SURFACES IN THE COMPLEX PROJECTIVE PLANE

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1. Settings

Let M be a smooth closed oriented 4-manifold (possibly with boundary) and F be a smooth closed oriented 2-manifold (possibly with boundary) embedded in M. An orientation preserving diffeomorphism ψ over F is *extendable* if there is an orientation preserving diffeomorphism Ψ over M such that $\Psi|_F = \psi$. In general, for an oriented manifold A and its submanifold B, we denote

$$\mathrm{Diff}_+(A, \mathrm{fix}\ B) = \left\{ \begin{array}{l} \text{orientation preserving diffeomorphisms } \psi \text{ over } A \\ \text{such that } \psi|_B = id_B \end{array} \right\}.$$

The group $\pi_0(\mathrm{Diff}_+(F, \mathrm{fix}\ \partial F))$ is called the *mapping class group* of F and denoted by $\mathcal{M}_F$. If F is a closed oriented surface of genus g, this group is denoted by $\mathcal{M}_g$. We define

$$\mathcal{E}(M, F) = \{\varphi \in \mathcal{M}_F \mid \varphi \text{ is extendable } \}.$$

This is a subgroup of $\mathcal{M}_F$ and is a central object of this note.

2. Surfaces in the 4-sphere

In this section, we review some known facts on $\mathcal{E}(S^4, F)$ for the surface embedded in S^4.

A *3-dimensional handlebody* H_g is an oriented 3-manifold which is constructed from a 3-ball with attaching g 1-handles. Any image of embeddings of H_g into S^4 are isotopic each other. Therefore, $(S^4, \partial H_g)$ is unique. A surface *standardly* (or *trivially*) *embedded* in S^4 is $(S^4, \partial H_g)$. In [10] (the case where $g = 1$) and [5](the case where $g = 2$), we showed:

Theorem 2.1. *An orientation preserving diffeomorphism ϕ on ∂H_g is extendable to S^4 if and only if ϕ preserves the Rokhlin quadratic from (i.e. the spin structure on ∂H_g induced from S^4).*

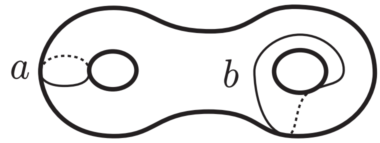

Figure 1

2000 *Mathematics Subject Classification.* 57Q45, 57N05, 20F38.
Key words and phrases. plane curve, knotted surface, mapping class group.

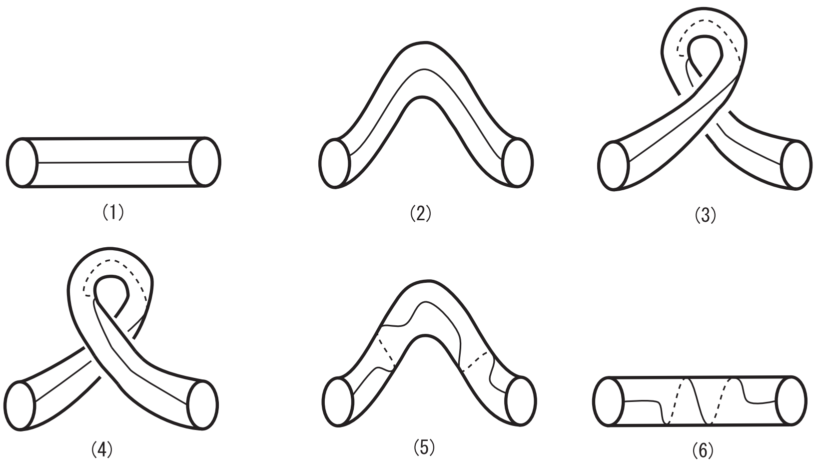

FIGURE 2

The subgroup $\mathcal{SP}_g$ of $\mathcal{M}_g$ which consists of the all elements preserving the Rokhlin quadratic form is called the (even) *spin mapping class group*. Figure 1 illustrates H_g in the equator S^3 of S^4, We denote the right hand Dehn twist about a simple closed curve c on Σ_g by the symbol T_c. Two maps T_a^2 and T_b (where a and b are as illustrated in Figure 1) are typical elements of $\mathcal{SP}_g$. Figure 2 indicates that $T_a^2 \in \mathcal{E}(S^4, \partial H_g)$. In the next section, we show $T_b \in \mathcal{E}(S^4, \partial H_g)$.

If (S^4, F) is not standard, there may be other obstruction to extend diffeomrphisms. Let (S^3, k) be a knot in S^3 and $(S^4, S(k))$ (resp. $(S^4, \tilde{S}(k))$) be the spin (resp. twisted-spin) of (S^3, k). When (S^3, k) is a torus knot, Iwase [7] investigated $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}(S^4, \tilde{S}(k))$, and when (S^3, k) is an arbitrary knot, the author [4] investigated these groups.

3. A Hopf band on the boundary of the 4-ball

A link L in S^3 is called a *fiber link* if there is a map $\phi : S^3 \setminus L \to S^1$ which is a fiber bundle projection. For each $t \in S^1$, $\phi^{-1}(t) = F$ does not depend on t and called the *fiber* of L. Since ϕ is a bundle projection, $S^3 \setminus L$ is diffeomorphic to the quotient of $F \times [0,1]$ by an equivalence $(x, 0) \sim (h(x), 1)$ where h is a diffeomorphism over F and called the *monodromy* of L.

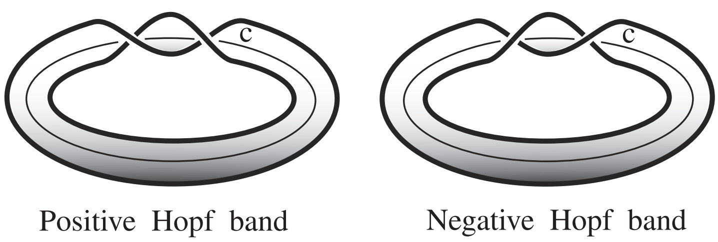

FIGURE 3

A *Hopf band* is an annulus embedded in S^3 as in Figure 3. In this picture, there are two types of Hopf bands. In this note, we treat both types of Hopf bands. The boundary of a Hopf band is called a *Hopf link*. The Hopf link is a fiberd link whose fiber is the Hopf band and whose monodromy is a Dehn twist about the core circle of the Hopf band. Let B be a Hopf band in S^3 which is a boundary of a 4-ball D^4. We push the interior of B into the interior of D^4 and let B' be the annulus obtained by this deformation and let c be the core circle of B'.

Proposition 3.1. *The Dehn twist T_c about c is extendable, i.e. there is an element $T \in \mathrm{Diff}_+(D^4, \mathrm{fix}\, \partial D^4)$ such that $T|_{B'} = T_c$.*

Proof . Since ∂B is a fiber link, whose fiber is B and whose monodromy is T_c, there is an orientation preserving diffeomorphism ψ of S^3 such that $\psi|_B = T_c$, and there is an isotopy ψ_t $(t \in [0,1])$ with $\psi_0 = id_{S^3}$ and $\psi_1 = \psi$, which is defined by shifting fibers. Let $N(\partial D^4)$ be the regular neighborhood of ∂D^4 in D^4. We parametrize $N(\partial D^4) = S^3 \times [0,2]$ so that $S^3 \times \{0\} = \partial D^4$ and $B' = \partial B \times [0,1] \cup B \times \{1\}$. Let T be a diffeomorphism defined as follows

$$T|_{N(\partial D^4)}(x,t) = \begin{cases} (\psi_t(x), t) & 0 \le t \le 1 \\ (\psi_{2-t}(x), t) & 1 \le t \le 2 \end{cases}$$

$$T|_{D^4 \setminus N(\partial D^4)} = id.$$

This is the diffeomorphism which we need. □

Here we explain why $T_b \in \mathcal{E}(S^4, \partial H_g)$. Let B be the regular neighborhood of b on ∂H_g. This B is a Hopf band in S^3. We apply Proposition 3.1 to B, then we conclude that $T_b \in \mathcal{E}(S^4, \partial H_g)$.

Remark 3.2. For each element ϕ of $\mathcal{E}(S^4, \partial H_g)$, we take its extension Φ to be an orientation preserving diffeomorphism isotopic to the identity on S^4.

4. Surfaces in the complex projective plane

For the free action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathbb{C}^3 \setminus \{(0,0,0)\}$ defined by $\lambda(z_0, z_1, z_2) = (\lambda z_0, \lambda z_1, \lambda z_2)$, we take the quotient $\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{(0,0,0)\})/\mathbb{C}^*$. This space $\mathbb{CP}^2$ is a closed oriented 4-manifold and called the *complex projective plane*. This 4-manifold $\mathbb{CP}^2$ is constructed from D^4 by attaching a 2-handle h^2 along the frame 1 trivial knot K_0 in ∂D^4, and attaching a 4-handle h^4. Any image of embeddings of the 3-dimensional handlebody H_g into $\mathbb{CP}^2$ are isotopic each other. Therefore, $(\mathbb{CP}^2, \partial H_g)$ is unique. A surface *standardly embedded* in $\mathbb{CP}^2$ is $(\mathbb{CP}^2, \partial H_g)$. We obtain:

Theorem 4.1. *For any g, $\mathcal{E}(\mathbb{CP}^2, \partial H_g) = \mathcal{M}_g$.*

Let C_d be a non-singular plane curve, then C_d is a genus $\frac{(d-1)(d-2)}{2}$ closed oriented surface embedded in $\mathbb{CP}^2$. We remark that C_d is unique up to isotopy, $C_d = \{[X : Y : Z] \in \mathbb{CP}^2 | X^d + Y^d + Z^d = 0\}$ and $[C_d] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. We obtain:

Theorem 4.2. *When $d = 3, 4$, $\mathcal{E}(\mathbb{CP}^2, C_d) = \mathcal{M}_{g_d}$, where $g_d = \frac{(d-1)(d-2)}{2}$.*

We use Proposition 3.1 for a proof of Theorems 4.1 and 4.2. For details, please consult [6].

Remark 4.3. For each element ϕ of $\mathcal{E}(\mathbb{CP}^2, \partial H_g)$ or $\mathcal{E}(\mathbb{CP}^2, C_d)$, we take its extension Φ to be an orientation preserving diffeomorphism isotopic to the identity on $\mathbb{CP}^2$.

When $d \geq 5$, $\mathcal{E}(\mathbb{CP}^2, C_d)$ is unknown. It is, however, not the case that $\mathcal{E}(\mathbb{CP}^2, C_d) = \mathcal{M}_{g_d}$, because, when d is odd, C_d is a characteristic surface, so Rokhlin quadratic form on $H_1(C_d; \mathbb{Z}_2)$ is well-defined (for the definition of Rokhlin quadratic form, see [11], [9] and [2]). By the definition of Rokhlin quadratic form, if a diffeomorphism on C_d is extendable to $\mathbb{CP}^2$, this diffeomorphism should preserve this form. Hence:

Theorem 4.4. *When d is an odd integer greater than or equal to 5, $\mathcal{E}(\mathbb{CP}^2, C_d)$ is proper subgroup of $\mathcal{M}_{g_d}$, where $g_d = \frac{(d-1)(d-2)}{2}$.*

Acknowledgments

The author would like to express his gratitude to Professors Masaharu Ishikawa, Masahico Saito, and Akira Yasuhara for fruitful discussions and comments.

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