Proceedings of the East Asian School
of Knots, Links, and Related Topics
February 16–20, 2004 • Seoul, Korea

THE SMALE CONJECTURE FOR LENS SPACES

SUNGBOK HONG, DARRYL MCCULLOUGH, AND J. H. RUBINSTEIN

Lens space means a 3-dimensional lens space $L(m, q)$ other than $L(1, 0)$ (the 3-sphere), $L(0, 1)$ (the product $S^2 \times S^1$), and *other than* $L(2, 1)$ (the real projective 3-space $\mathbb{RP}^3$).

Lens spaces are elliptic 3-manifolds. That is, they may be regarded as the quotient of the standard round 3-sphere S^3 by a finite subgroup of the group SO(4) of orientation-preserving isometries of S^3. Therefore, they inherit Riemannian metrics of constant positive curvature.

S. Smale proved that for the standard round 2-sphere S^2, the inclusion of the isometry group O(3) into the diffeomorphism group $\mathrm{Diff}(S^2)$ is a homotopy equivalence. He conjectured that the same is true for the 3-sphere, that is, that $\mathrm{O}(4) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence.

J. Cerf proved that the inclusion induces a bijection on path components, and the full conjecture was proven by A. Hatcher.

It is a result of fundamental importance in the theory of 3-manifolds, because it shows that smooth structures on 3-manifolds are unique, and that there is no essential difference between the group of homeomorphisms and the group of diffeomorphisms of a 3-manifold (due to J. Cerf).

A natural extension of this conjecture is that if M is any elliptic 3-manifold, then $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence. This has been called the Generalized Smale Conjecture. In this talk, we will discuss the Smale Conjecture for Lens Spaces:

Theorem 1. *For any lens space L, the inclusion $\mathrm{Isom}(L) \to \mathrm{Diff}(L)$ is a homotopy equivalence.*

D. McCullough [2002] proved the "π_0-part" of the Smale Conjecture for elliptic 3-manifolds.

Theorem 2. *Let M be an elliptic 3-manifold. Then the inclusion of $\mathrm{Isom}(M)$ into $\mathrm{Diff}(M)$ is a bijection on path components.*

Consequently, to prove the Smale Conjecture for a lens space L, it is sufficient to prove that the inclusion of the connected components of the identity map $\mathrm{isom}(L) \to \mathrm{diff}(L)$ is a homotopy equivalence.

Palais showed that $\mathrm{Diff}(L)$ has the homotopy type of a a CW complex so by Whitehead theorem it is sufficient to show that $\pi_q \,\mathrm{isom}(L) \to \pi_q \,\mathrm{diff}(L)$ is an isomorphism for every $q \geq 1$. Namely, $\mathrm{isom}(L)$ and $\mathrm{diff}(L)$ are weak homotopy equivalent.

Received by the editors on February 26, 2004.

2000 *Mathematics Subject Classification.* Primary 57M99; Secondary 57M50, 57R50, 57S05, 58D05.

Partially supported by the Korea Research Foundation.

Partially supported by the National Science Foundation.

A diffeomorphism from L to L is called *fiber-preserving* if the image of each fiber is a fiber, and *vertical* if it preserves each fiber.

D. McCullough gives a certain way to imbed $\pi_1(L)$ into $\mathrm{isom}(S^3) = \mathrm{SO}(4)$ so that its action on S^3 is fiber-preserving for the fibers of the Hopf bundle structure of S^3. Consequently, this bundle structure descends to a Seifert fibering of L, which we call the *Hopf fibering* of L.

If $q = 1$, this Hopf fibering is actually an S^1-bundle structure, while if $q > 1$, it has two exceptional fibers with invariants of the form (k, q_1), (k, q_2) where $k = m/\gcd(q-1, m)$. We will always use the Hopf fibering as the Seifert-fibered structure of L.

By $\mathrm{diff}_f(L)$ we denote the connected component of the identity map in the space of fiber-preserving diffeomorphisms.

D.McCullough shows that if $m > 2$, then every orientation-preserving isometry of L preserves the Hopf fibering on L. In particular, $\mathrm{isom}(L) \subset \mathrm{diff}_f(L)$, so there are inclusions

$$\mathrm{isom}(L) \to \mathrm{diff}_f(L) \to \mathrm{diff}(L) \ .$$

Theorem 3. *If* $m > 2$, *then the inclusion* $\mathrm{isom}(L) \to \mathrm{diff}_f(L)$ *is a weak homotopy equivalence.*

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & \mathrm{isom}(L) \arrow[r] \arrow[d] & \mathrm{isom}(L_0) \arrow[d] \\
\mathrm{vert}(L) \arrow[r] & \mathrm{diff}_f(L) \arrow[r] & \mathrm{diff}_{orb}(L_0)
\end{tikzcd}$$

where L_0 is the quotient orbifold and $\mathrm{diff}_{orb}(L_0)$ is the group of orbifold diffeomorphisms of L_0, and $\mathrm{vert}(L)$ is the group of vertical diffeomorphisms.

The above theorem reduces the Smale Conjecture for lens spaces to proving that the inclusion $\mathrm{diff}_f(L) \to \mathrm{diff}(L)$ is a weak homotopy equivalence.

For this, we need to prove that for all $k \geq 1$, $\pi_k(\mathrm{diff}_f(L))$ and $\pi_k(\mathrm{diff}(L))$ are isomorphic.

An element of $\pi_k \,\mathrm{diff}(L)$ can be represented as a map $f\colon S^k \to \mathrm{diff}(L)$. Namely, a continuously parametrized family of diffeomorphisms from L to itself. This also can be regarded as a map $f\colon S^k \times L \to L$. It suffices to continuously deform this family a family for which each $f_t = f|_{\{t\}\times L}\colon L \to L$ is a fiber preserving diffeomorphism. This shows that $\pi_k \,\mathrm{diff}_f(L) \to \pi_k \,\mathrm{diff}(L)$ is onto. A combination of tricks and minor modifications can be used to show it is injective.

A major difficulty working with elliptic 3-manifolds is their lack of incompressible surfaces. In their place, we use another structure which has a certain degree of essentiality, called a *sweepout.*

This means a structure on L as a quotient of $P \times I$, where P is a torus, in which $P \times \{0\}$ and $P \times \{1\}$ are collapsed to core circles of the tori of a genus 1 Heegaard splitting of L. For $0 < t < 1$, $P \times \{t\}$ becomes a Heegaard torus in L, denoted by P_t and called a *level.* The sweepout is chosen so that each P_t is a union of fibers.

Start with a parameterized family $f\colon L \times S^k \to L$, and $u \in S^k$ denote by f_u the restriction of f to $L \times \{u\}$. According to the reductions, we need to deform f so that each f_u is fiber-preserving. There are three major steps to the deformation procedure.

Step 1 ("finding good levels") is to perturb the family f so that for each u, there is some pair (s, t) so that $f_u(P_s)$ intersects P_t transversely, in a collection of circles each of which is either essential in both $f_u(P_s)$ and P_t (a *biessential* intersection), or inessential in both (a *discal* intersection), and at least one intersection is biessential.

This pair is said to intersect in *good position,* and if none of the intersection is discal, in *very good position.*

To accomplish Step 1, the methodology of Rubinstein-Scharlemann is adapted. First, one perturbs f to be in *general position,* as defined in later transcript. This means that the tangencies of intersections of $f(P_s)$ and P_t are sufficiently well-controlled to define a *graphic* in the square I^2.

That is, the pairs (s, t) for which $f(P_s)$ and P_t do not intersect transversely form a graph imbedded in the square. The complementary regions of the graphic correspond to pairs intersecting transversely. They are labeled according to a procedure in Rubinstein-Scharlemann's paper.

We show that the properties of general position salvage enough of the combinatorics of these labels to deduce that at least one of the complementary regions consists of pairs in good position.

Then by using some very nice results of J.W Bruce, we deduce that a family can be perturbed into general position,

Step 2 ("from good to very good") is to deform f to eliminate the discal intersections of $f_u(P_s)$ and P_t, so that they intersect in very good position.

This is an application of Hatcher's parameterization methods. One must be careful here, since an isotopy that eliminates a discal intersection can also eliminate a biessential intersection, and if all biessential intersections were eliminated by the procedure, the resulting pair would not be in very good position. A careful observation ensures that not all biessential intersections will be eliminated.

Step 3 ("from very good to fiber-preserving") is to use the pairs in very good position to deform the family so that each f_u is fiber-preserving.

The basic ideas is first to use the biessential intersections to deform the f_u so that $f_u(P_s)$ actually equals P_t, then use the contractibility of the components of the diffeomorphism groups (relative to the boundary) of surfaces and Haken 3-manifolds to make the f_u fiber-preserving on P_s and then on its complementary solid tori.

This process is technically complicated for two reasons. First, although a biessential intersection is essential in both tori, it can be contractible in one of the complementary solid tori of P_t, and $f_u(P_s)$ can meet that complementary torus in annuli that are not parallel into P_t. So one may be able to push the annuli out from only one side of P_t.

Secondly, the fitting together of these isotopies requires one to work with not just one level but levels at a single parameter.

A *sweepout* of a closed orientable 3-manifold is a smooth map $\sigma\colon F \times [0, 1] \to M$, where F is a closed connected surface, such that

1. $S_0 = \sigma(F \times \{0\})$ and $S_1 = \sigma(F \times \{1\})$ are disjoint graphs of valence 3 in M,
2. Each S_i is a union of a collection of smoothly imbedded arcs and circles in M.

(3) $\sigma|_{F\times(0,1)}\colon F\times(0,1)\to M$ is a diffeomorphism onto $M-(S_0\cup S_1)$,
(4) Near $F\times\partial I$, σ gives a mapping cylinder neighborhod of $S_0\cup S_1$.

Associated to any t with $0<t<1$, there is a Heegaard splitting $M=V_t\cup W_t$, where $V_t=\sigma(F\times[0,t])$ and $W_t=\sigma(F\times[t,1])$. For each t, S_0 is a deformation retract of V_t and S_1 is a deformation retract of W_t.

We denote $\sigma(F\times\{t\})$ by P_t, and call it a *level surface* for σ. Also, for $s,t\in(0,1)$ with $s\neq t$, we denote the closure of the region between P_s and P_t by $R(s,t)$.

Note that any genus-1 Heegaard splitting of L provides sweepouts with S_0 and S_1 as core circles of the two solid tori.

The next lemma gives very strong restrictions on level tori of two different sweepouts of a lens space that intersect in very good position. For its proof, recall that a *spine* for a closed surface F is a 1-dimensional cell complex in F whose complement consists of open disks.

Lemma 4. *Let L be a lens space. Let $\sigma\colon T\times[0,1]\to L$ be a sweepout as above, where T is a torus. Let $\tau\colon T\times[0,1]\to L$ be another sweepout, with level surfaces $Q_s=\tau(T\times\{s\})$. Suppose that for $t_1<t_2$, $s_1\neq s_2$, and $i=1,2$, Q_{s_i} and P_{t_i} intersect in very good position, and that Q_{s_1} has no discal intersections with P_{t_2}. If Q_{s_1} has nonempty intersection with P_{t_2}, then either*

(1) *every intersection circle of Q_{s_1} with P_{t_2} is biessential, and consequently $Q_{s_1}\cap R(t_1,t_2)$ contains an annulus with one boundary circle essential in P_{t_1} and the other essential in P_{t_2}, or*
(2) *for $i=1,2$, $Q_{s_i}\cap P_{t_i}$ consists of meridians of W_{t_i}, and $Q_{s_1}\cap R(t_1,t_2)$ contains a surface Σ which is a homology from a circle of $Q_{s_1}\cap P_{t_1}$ to a union of circles in P_{t_2}.*

Consider a smooth function $f\colon(\mathbb{R}^2,0)\to(\mathbb{R},0)$. We say that a critical point of f is *stable* when it is locally equivalent under smooth change of coordinates of the domain and range to $f(x,y)=x^2+y^2$ or $f(x,y)=x^2-y^2$. The first type is called a center , and the second a saddle. An unstable critical point is called a birth-death point if it is locally $f(x,y)=x^2+y^3$.

Let $\sigma,\tau\colon F\times[0,1]\to L$ be sweepouts. Denote $\sigma(F\times\{0,1\})$ by S, $\tau(F\times\{0,1\})$ by T, $\sigma(F\times\{t\})$ by P_t, and $\tau(F\times\{t\})$ by Q_t.

We call P_t a σ-level and Q_t a τ-level. A tangency of Q_s and P_t at a point w is said to be *of Morse type* at w if in some local xyz-coordinates with origin at w, P_t is the xy-plane and Q_s is the graph of a function which has a stable critical point or a birth-death point at the origin. Note that this condition is symmetric in P_t and Q_s.

(1) If Q_s contains a precompression for P_t in V_t (respectively, in W_t), the region receives the letter A (respectively, B).
(2) If P_t contains a precompression for Q_s in X_s (respectively, in Y_s), the region receives the letter X (respectively, Y).
(3) If the region has neither an A-label nor a B-label, and V_t (respectively, W_t), contains a spine of Q_s, the region receives the letter b (respectively, a).
(4) If the region has neither an X-label nor a Y-label, and X_s (respectively, Y_s), contains a spine of P_t, the region receives the letter y (respectively, x).

We call the data consisting of the graph $\Gamma \subset I^2$ and the labeling of a subset of its complementary regions the *Rubinstein-Scharlemann graphic* associated to the sweepouts.

We say that f is *in general position* (with respect to the sweepout τ) if there exists $\epsilon > 0$ such that ϵ gives border label control for f and such that the following hold for each parameter $u \in W$.

(GP1) For each (s, t) in I^2_ϵ, $Q_s \cap P_t$ is a graph. At each point in an open edge of this graph, Q_s meets P_t transversely. At each vertex, they are tangent.

(GP2) The $(s, t) \in I^2_\epsilon$ for which Q_s has a tangency with P_t form a graph Γ_u in I^2_ϵ.

(GP3) If (s, t) lies in an open edge of Γ_u, then Q_s and P_t have a single stable tangency.

Here is the main result about deforming f so that it can be put into general position with respect to τ.

Theorem 5. *Let $f\colon M \times W \to M$ be a parameterized family of diffeomorphisms. Then by an arbitrarily small deformation, f can be put into general position with respect to τ.*

We adapt the methodology of Bruce to prove the above theorem.

Theorem 6 (J. W. Bruce)**.** *Let A, B and U be smooth manifolds and $C \subset B$ a submanifold. There is a residual family of mappings $F \in C^\infty(A \times U, B)$ such that:*

(a) *For each $u \in U$, the restriction $F_u = F|_{A\times\{u\}}\colon A \to B$ is transverse to C except possibly on a discrete set of points.*

(b) *For each $u \in U$, the set $F_u^{-1}(C)$ is a smooth transverse preimage of C, except possibly on a discrete set of points. At each of these exceptional points $F_u^{-1}(C)$ is locally diffeomorphic to the germ of an algebraic variety, with the exceptional point corresponding to an isolated singular point of the variety.*

The basic idea of the proof is to define a locally algebraic subset of a jet space which contains all the maps that fail these weak transversality conditions.

These subsets have increasing codimension as higher-order jets are taken. A variant of Thom transversality then allows one to perturb a parameterized family of maps so that these jets are avoided and the conclusion holds.

When A and W are compact, the image of $A \times W$ will lie in the open complement of the locally algebraic sets of sufficiently high codimension. Consequently, any map sufficiently close to the perturbed map will also satisfy the conclusions of the theorem.

Theorem 7. *Suppose that $f\colon L \times W \to L$ is in general position with respect to τ. Then for each u there exists (s, t) such that Q_s meets P_t in good position.*

Lemma 8. *Let X be either a solid torus or $S^1 \times S^1 \times I$, with a fixed Seifert fibering. Then the inclusion $\operatorname{diff}_f(X) \to \operatorname{diff}(X)$ is a homotopy equivalence.*

Here is a very brief sketch of the proof.

Results from surface theory imply that $\operatorname{diff}(S^1 \times S^1) \simeq S^1 \times S^1$. Using the results of J. Kalliongis and D.McCullough, $\operatorname{diff}_f(X)$ is homotopy equivalent to $S^1 \times S^1$, and if T is a boundary component, the restriction map $\operatorname{diff}_f(X) \to \operatorname{diff}(T)$ is a weak homotopy equivalence. Using Hatcher's results, one can show that the fiber of $\mathrm{diff}(X) \to \mathrm{diff}(T)$ is contractible, so in $\mathrm{diff}_f(X) \to \mathrm{diff}(X) \to \mathrm{diff}(T)$, the composition and the second map are homotopy equivalences, hence the first is as well.

The Q_{s_i} meet the P_{t_i} in very good position. Their intersection circles of $Q_{s_i} \cap P_{t_i}$ cannot be meridians in both V_{t_i} and W_{t_i}. So Q_{s_i} must satisfy exactly one of the following:

1. The circles of $Q_{s_i} \cap P_{t_i}$ are not longitudes or meridians for V_{t_i}, so the annuli of $Q_{s_i} \cap V_{t_i}$ are uniquely boundary parallel in V_{t_i}.
2. The circles of $Q_{s_i} \cap P_{t_i}$ are longitudes or meridians for V_{t_i}, but are not longitudes or meridians for W_{t_i}, so the annuli of $Q_{s_i} \cap W_{t_i}$ are uniquely boundary parallel in W_{t_i}.
3. The circles of $Q_{s_i} \cap P_{t_i}$ are longitudes both for V_{t_i} and for W_{t_i}.

Theorem 9. *Let $L = L(m,q)$ with $m > 2$ and let $\phi\colon S^d \to \mathrm{diff}(L)$. Then ϕ is deformable to a family of fiber-preserving diffeomorphisms.*

DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
SEOUL 136-701
KOREA
E-mail address: `shong@semi.korea.ac.kr`
URL: `http://elie.korea.ac.kr/~shong/`

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019
U. S. A.
E-mail address: `dmccullough@math.ou.edu`
URL: `www.math.ou.edu/~dmccullo/`

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA 3052
AUSTRALIA
E-mail address: `rubin@ms.unimelb.edu.au`
URL: `www.ms.unimelb.edu.au/~rubin/`