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SKEIN RELATIONS OF THE LINKS–GOULD INVARIANT AND THEIR APPLICATIONS

ATSUSHI ISHII

Abstract. We introduce two skein relations. One is obtained by deforming the known cubic skein relation. Another one is obtained from the first skein relation by replacing a tangle with similar one.

Although it is not easy to evaluate the Links–Gould invariant without the aid of a computer because of the size of the R-matrix, these skein relations lead us to recursive calculation of the invariant for Conway's algebraic links.

By using these skein relations we obtain a formula for the Links–Gould invariant of the family of knots given by Kanenobu, which include infinitely many knots with the same Jones and HOMFLY polynomials, and show that the invariant is complete for this family.

1. Introduction

The Links–Gould invariant, or the LG invariant for short, is a two-variable polynomial invariant of oriented links. The LG invariant was derived from the one-parameter family of four dimensional representations of the quantum superalgebra $U_q[gl(2|1)]$ [9]. D. De Wit, L. H. Kauffman and J. R. Links [3] gave the explicit form of the R-matrix and showed that the LG invariant is powerful through evaluation of some links. In particular, D. De Wit [2] showed that the invariant is complete for all prime knots of up to 10 crossings.

We give two skein relations which lead us to recursive calculations of the invariant for algebraic links. By using the skein relations, we give a formula for the LG invariant of the family of knots given by Kanenobu, which include infinitely many knots with the same Jones and HOMFLY polynomials, and show that the invariant is complete for this family.

2. Preliminaries

Any oriented tangle diagram can be expressed up to isotopy as a diagram composed from the elementary tangle diagrams shown in Figure 1. Furthermore any oriented tangle diagram can be expressed up to isotopy as a *sliced diagram* which is such a diagram sliced by horizontal lines such that each domain between adjacent horizontal lines has either a single crossing or a single critical point.

Let V be a vector space and let V^* be the dual space of it. We consider an invertible endomorphism $R : V \otimes V \to V \otimes V$ and linear maps $n : V \otimes V^* \to \mathbb{C}$, n' :

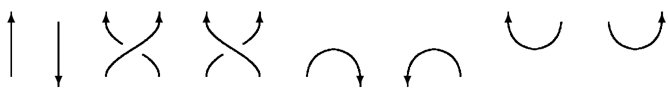


Figure 1

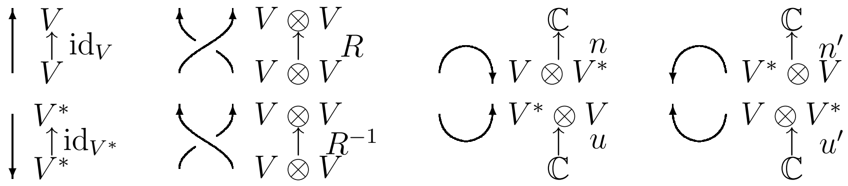
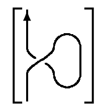


FIGURE 2

$V^* \otimes V \to \mathbb{C}$, $u : \mathbb{C} \to V^* \otimes V$, and $u' : \mathbb{C} \to V \otimes V^*$; and we associate these maps to elementary oriented tangle diagrams as described in Figure 2. Corresponding to any oriented tangle diagram D, we may then obtain a linear map $[D]$ as the composition of tensor products of copies of the linear maps associated to the elementary tangle diagrams within D. For example,



$$= (\mathrm{id}_V \otimes n)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes u').$$

The Links–Gould invariant is defined as follows. Let V be a four dimensional vector space with a basis $\{e_i\}_{i=1}^4$, and dual basis $\{e_i^*\}_{i=1}^4$. We denote by $e_{j_1 \cdots j_n}^{i_1 \cdots i_n}$ the linear map $V^{\otimes n} \to V^{\otimes n}$ defined by

$$e_{j_1 \cdots j_n}^{i_1 \cdots i_n}(e_{k_1} \otimes \cdots \otimes e_{k_n}) = \delta_{k_1}^{j_1} \cdots \delta_{k_n}^{j_n} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

where δ_j^i is the Kronecker symbol. In the same way we define the linear maps $e_{i_1 \cdots i_n} : W_1 \otimes \cdots \otimes W_n \to \mathbb{C}$ and $e^{i_1 \cdots i_n} : \mathbb{C} \to W_1 \otimes \cdots \otimes W_n$, where each W_k is either V or V^*. For example, $e_{i_1 i_2} : V \otimes V \to \mathbb{C}$ and $e^{i_1 i_2} : \mathbb{C} \to V \otimes V^*$ are defined by $e_{i_1 i_2}(e_{k_1} \otimes e_{k_2}) = \delta_{k_1}^{i_1} \delta_{k_2}^{i_2}$ and $e^{i_1 i_2}(1) = e_{i_1} \otimes e_{i_2}^*$ respectively.

We obtain the bracket [ ] by putting

$$\begin{aligned}
R &= t_0 e_{11}^{11} - \left(e_{22}^{22} + e_{33}^{33}\right) + t_1 e_{44}^{44} + (t_0 - 1)\left(e_{21}^{21} + e_{31}^{31}\right) + (t_0 - 1)(1 - t_1) e_{41}^{41} \\
&\quad + (t_1 - 1)\left(e_{42}^{42} + e_{43}^{43}\right) + (t_0 t_1 - 1) e_{23}^{23} + \left(e_{41}^{14} + e_{14}^{41}\right) - t_0^{1/2} t_1^{1/2}\left(e_{32}^{23} + e_{23}^{32}\right) \\
&\quad + t_0^{1/2}\left(e_{21}^{12} + e_{12}^{21} + e_{31}^{13} + e_{13}^{31}\right) + t_1^{1/2}\left(e_{42}^{24} + e_{24}^{42} + e_{43}^{34} + e_{34}^{43}\right) \\
&\quad - t_0^{1/2} t_1^{1/2} \left((t_0 - 1)(1 - t_1)\right)^{1/2}\left(e_{41}^{23} + e_{23}^{41}\right) + \left((t_0 - 1)(1 - t_1)\right)^{1/2}\left(e_{41}^{32} + e_{32}^{41}\right), \\
n &= e_{11} + e_{22} + e_{33} + e_{44}, \\
n' &= t_0 e_{11} - t_1^{-1} e_{22} - t_0 e_{33} + t_1^{-1} e_{44}, \\
u &= e^{11} + e^{22} + e^{33} + e^{44}, \\
u' &= t_0^{-1} e^{11} - t_1 e^{22} - t_0^{-1} e^{33} + t_1 e^{44}.
\end{aligned}$$

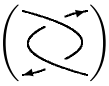
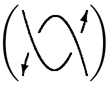
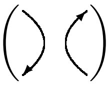
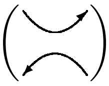
For a $(1,1)$-tangle T, the Links–Gould invariant of the link $\widehat{T}$, which is the closure of T, is defined by the following identity:

$$[D_T] = LG(\widehat{T}; t_0, t_1)\mathrm{id}_V,$$

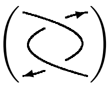
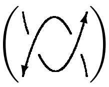
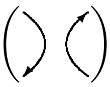
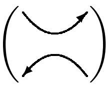
where D_T is a tangle diagram of T. Note that $LG(L; p^{-2}, p^2 q^2)$ (where $p = q^\alpha$) coincides with the Links–Gould invariant in [3], where α originates as a complex parameter of a family of $U_q[gl(2|1)]$ representations. For the details we refer the reader to [3, 10].

3. Skein relations

Theorem 1 ([4])**.** *The LG invariant satisfies the following skein relations.*
Skein relation (i):

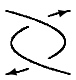
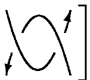
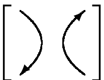
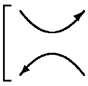
$LG\Big($  $\Big) + t_0t_1 LG\Big($  $\Big) = (c+1)LG\Big($  $\Big) + (c+t_0t_1)LG\Big($  $\Big)$,

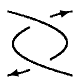
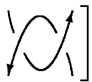
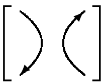
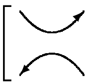
Skein relation (ii):

$LG\Big($  $\Big) - LG\Big($  $\Big) = (1-c)LG\Big($  $\Big) + (c-1)LG\Big($  $\Big)$,

where $c = -(t_0-1)(t_1-1)$.

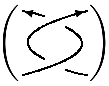
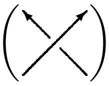
Proof. The following equalities are verified with aid of computer:

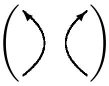
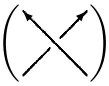
$\Big[$  $\Big] + t_0t_1 \Big[$  $\Big] = (c+1)LG\Big[$  $\Big] + (c+t_0t_1)LG\Big[$  $\Big]$,

$\Big[$  $\Big] - \Big[$  $\Big] = (1-c)LG\Big[$  $\Big] + (c-1) \Big[$  $\Big]$.

□

Remark. In [3, p170], the following skein relation is derived from the identity $(R - t_0\mathrm{id}_V)(R - t_1\mathrm{id}_V)(R + \mathrm{id}_V) = 0$ (see also [9, p192]).
Skein relation (o):

$LG\Big($  $\Big) + (1-t_0-t_1)LG\Big($  $\Big)$

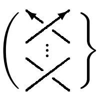
$+(t_0t_1 - t_0 - t_1)LG\Big($  $\Big) + t_0t_1 LG\Big($  $\Big) = 0.$

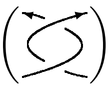
By "adding" the tangle  to Skein relation (o) from the left, we obtain Skein relation (i). (Therefore Skein relation (o) is equivalent to Skein relation (i).)

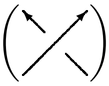
Theorem 2 ([4])**.** *We can evaluate the LG invariant of Conway's algebraic links recursively.*

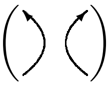
For the definition of Conway's algebraic links, we refer the reader to [1]. We remark that the links include 2-bridge links.
The following formulas follow from Theorem 1.

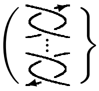
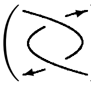
Corollary 3 ([4])**.** *The LG invariant satisfies the following skein relations.*

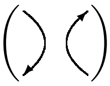
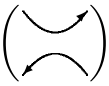
$LG\Big($  $\Big\}$ *n half twists*$\Big)$

$$= \left(\frac{(-1)^n}{(t_0+1)(t_1+1)} + \frac{t_0^n}{(t_0+1)(t_0-t_1)} + \frac{t_1^n}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big)$

$$- \left(\frac{(-1)^n(t_0+t_1)}{(t_0+1)(t_1+1)} + \frac{t_0^n(t_1-1)}{(t_0+1)(t_0-t_1)} + \frac{t_1^n(t_0-1)}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big)$

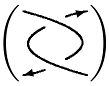
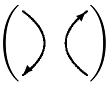
$$+ \left(\frac{t_0t_1(-1)^n}{(t_0+1)(t_1+1)} - \frac{t_1t_0^n}{(t_0+1)(t_0-t_1)} - \frac{t_0t_1^n}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big)$,

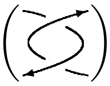
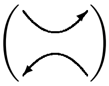
$LG\Big($  $\Big\}\, n\ \textit{full twists}\Big) = \dfrac{t_0^n t_1^n - 1}{t_0 t_1 - 1} LG\Big($  $\Big)$

$+\left(1 - \dfrac{t_0^n t_1^n - 1}{t_0 t_1 - 1}\right) LG\Big($  $\Big) - \dfrac{2c}{t_0 t_1 - 1}\left(n - \dfrac{t_0^n t_1^n - 1}{t_0 t_1 - 1}\right) LG\Big($  $\Big)$,

where $c = -(t_0 - 1)(t_1 - 1)$.

Proof. ¿From Skein relation (o), we obtain the first skein relation by induction. We obtain the following skein relation by subtracting Skein relation (ii) from Skein relation (i) after a half-rotation in a diagonal axis of tangles.

$LG\Big($  $\Big) - (t_0 t_1 + 1) LG\Big($  $\Big)$

$+ t_0 t_1 LG\Big($  $\Big) + 2(t_0 - 1)(t_1 - 1) LG\Big($  $\Big) = 0,$

¿From this skein relation, we obtain the second skein relation by induction. □

At the end of this section, we recall the following fundamental properties (see, for example, [4]).

Proposition 4. *The LG invariant has the following properties.*

(1) $LG(\bigcirc) = 1$ *for the trivial knot* $\bigcirc$.
(2) *The LG invariant of a split link vanishes.*
(3) $LG(L\#L') = LG(L)LG(L')$, *where* $\#$ *denotes a connected sum of* L *and* L'.
(4) $LG(L^*; t_0, t_1) = LG(L; t_0^{-1}, t_1^{-1})$, *where* L^* *denotes the reflection of* L.
(5) *The LG invariant does not detect inversion, which implies the symmetry* $LG(L; t_0, t_1) = LG(L; t_1, t_0)$.

4. Kanenobu knots

T. Kanenobu [7, 8] introduced a family of knots $K_{n,m}$ for $n, m \in \mathbb{Z}$, as in Figure 3. We also consider the knots $K_{n,\infty}$ and $K_{\infty,m}$ as in Figure 3. This family

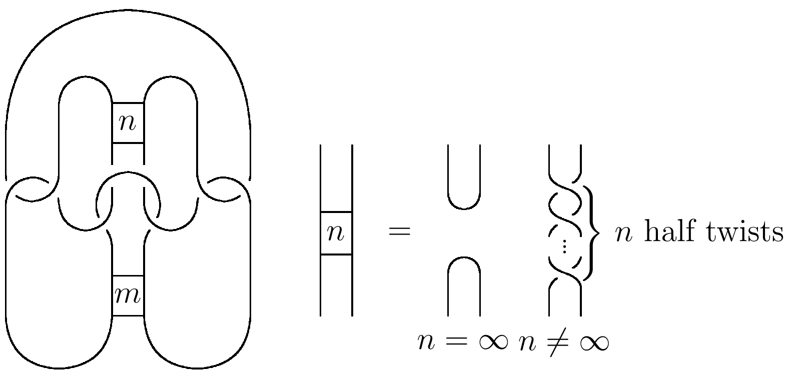


Figure 3

include infinitely many knots with the same HOMFLY polynomial and, therefore, the same Alexander polynomial and the same Jones polynomial [8, Theorem 3]. The family also include infinitely many chiral pairs which cannot be distinguished by the

Q polynomials. Z. Iwase and H. Kiyoshi completely classified the knots by using Jones polynomial and Q polynomial [6, Proposition 5], that is, $K_{n,m} = K_{n',m'}$ if and only if $\{n, m\} = \{n', m'\}$ for $n, m \in \mathbb{Z}$.

5. Formulas

We give a formula for the LG invariant of the Kanenobu knots, and show that the invariant is complete for this family of knots. As noted in the previous section, $K_{n,m} = K_{n',m'}$ if and only if $\{n, m\} = \{n', m'\}$ for $n, m \in \mathbb{Z}$. We remark that if n or m is ∞ then $K_{n,m}$ is the trivial 2-component link. This remark and Corollary 3 lead us to the following formula.

Theorem 5 ([5]). *For $i, j \in \mathbb{Z}$, $\epsilon, \delta = 0, 1$,*

$$LG(K_{2i-\epsilon,2j-\delta}; t_0, t_1)$$

$$= a(i)a(j)LG(K_{2-\epsilon,2-\delta}; t_0, t_1) + a(i)(1 - a(j))LG(K_{2-\epsilon,-\delta}; t_0, t_1)$$

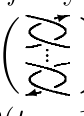
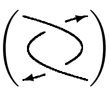
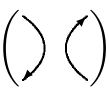
$$+(1 - a(i))a(j)LG(K_{-\epsilon,2-\delta}; t_0, t_1) + (1 - a(i))(1 - a(j))LG(K_{-\epsilon,-\delta}; t_0, t_1),$$

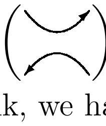
where $a(k) = (t_0^k t_1^k - 1)/(t_0t_1 - 1)$. For $-1 \le n, m \le 2$, $LG(K_{n,m}; t_0, t_1)$ is given by

$$\begin{aligned}
LG(K_{0,0}; t_0, t_1) &= (t_0t_1 - t_0 - t_1 + 2)^2(2t_0t_1 - t_0 - t_1 + 1)^2 t_0^{-2} t_1^{-2},\\
LG(K_{0,1}; t_0, t_1) &= LG(K_{1,0}; t_0, t_1)\\
&= -2t_0^3t_1^2 - 2t_0^2t_1^3 + 4t_0^3t_1 + 12t_0^2t_1^2 + 4t_0t_1^3 - 3t_0^3 - 24t_0^2t_1 - 24t_0t_1^2 - 3t_1^3\\
&\quad + t_0^3t_1^{-1} + 23t_0^2 + 54t_0t_1 + t_0^{-1}t_1^3 + 23t_1^2 - 12t_0^2t_1^{-1} - 60t_0 - 12t_0^{-1}t_1^2\\
&\quad - 60t_1 + 3t_0^2t_1^{-2} + 35t_0t_1^{-1} + 35t_0^{-1}t_1 + 3t_0^{-2}t_1^2 - 9t_0t_1^{-2} - 42t_0^{-1} - 9t_1t_0^{-2}\\
&\quad - 42t_1^{-1} + 22t_0^{-1}t_1^{-1} + 10t_0^{-2} + 10t_1^{-2} - 4t_0^{-1}t_1^{-2} - 4t_0^{-2}t_1^{-1} + 73,\\
LG(K_{0,-1}; t_0, t_1) &= LG(K_{-1,0}; t_0, t_1) = LG(K_{0,1}; t_0^{-1}, t_1^{-1}),\\
LG(K_{0,2}; t_0, t_1) &= LG(K_{2,0}; t_0, t_1)\\
&= (t_0t_1 - t_0 - t_1 + 2)(2t_0t_1 - t_0 - t_1 + 1)(2t_0^3t_1^3 - 2t_0^3t_1^2 - 2t_0^2t_1^3 + 4t_0^2t_1^2\\
&\quad - 3t_0^2t_1 - 3t_0t_1^2 + t_0^2 + 5t_0t_1 + t_1^2 - t_0 - t_1)t_0^{-2}t_1^{-2},
\end{aligned}$$

$$\begin{aligned}
&LG(K_{1,1};t_0,t_1)\\
&= t_0^4t_1^2+2t_0^3t_1^3+t_0^2t_1^4-t_0^4t_1-8t_0^3t_1^2-8t_0^2t_1^3-t_0t_1^4+11t_0^3t_1+24t_0^2t_1^2\\
&\quad +11t_0t_1^3-8t_0^3-34t_0^2t_1-34t_0t_1^2-8t_1^3+5t_0^3t_1^{-1}+29t_0^2+50t_0t_1\\
&\quad +5t_0^{-1}t_1^3+29t_1^2-3t_0^3t_1^{-2}-20t_0^2t_1^{-1}-45t_0-20t_0^{-1}t_1^2-3t_0^{-2}t_1^3-45t_1\\
&\quad +t_0^3t_1^{-3}+11t_0^2t_1^{-2}+30t_0t_1^{-1}+30t_0^{-1}t_1+11t_0^{-2}t_1^2+t_0^{-3}t_1^3-3t_0^2t_1^{-3}\\
&\quad -14t_0t_1^{-2}-19t_0^{-1}-14t_0^{-2}t_1-3t_0^{-3}t_1^2-19t_1^{-1}+3t_0t_1^{-3}+4t_0^{-1}t_1^{-1}\\
&\quad +6t_0^{-2}+3t_0^{-3}t_1+6t_1^{-2}-t_0^{-3}-t_1^{-3}+39,\\
&LG(K_{1,-1};t_0,t_1) = LG(K_{-1,1};t_0,t_1)\\
&= (2t_0^3t_1^3-2t_0^3t_1^2-2t_0^2t_1^3+2t_0^3t_1+4t_0^2t_1^2+2t_0t_1^3-t_0^3-3t_0^2t_1-3t_0t_1^2-t_1^3\\
&\quad +t_0^2+t_0t_1+t_1^2)(t_0^3t_1+t_0^2t_1^2+t_0t_1^3-t_0^3-3t_0^2t_1-3t_0t_1^2-t_1^3+2t_0^2+4t_0t_1\\
&\quad +2t_1^2-2t_0-2t_1+2)t_0^{-3}t_1^{-3},\\
&LG(K_{-1,-1};t_0,t_1) = LG(K_{1,1};t_0^{-1},t_1^{-1}),\\
&LG(K_{2,-1};t_0,t_1) = LG(K_{-1,2};t_0,t_1)\\
&= -3t_0^3t_1^2-3t_0^2t_1^3+5t_0^3t_1+16t_0^2t_1^2+5t_0t_1^3-2t_0^3-27t_0^2t_1-27t_0t_1^2-2t_1^3\\
&\quad +21t_0^2+56t_0t_1+21t_1^2-10t_0^2t_1^{-1}-58t_0-10t_0^{-1}t_1^2-58t_1+3t_0^2t_1^{-2}\\
&\quad +33t_0t_1^{-1}+33t_0^{-1}t_1+3t_0^{-2}t_1^2-10t_0t_1^{-2}-39t_0^{-1}-10t_0^{-2}t_1-39t_1^{-1}\\
&\quad +t_0t_1^{-3}+20t_0^{-1}t_1^{-1}+12t_0^{-2}+t_0^{-3}t_1+12t_1^{-2}-5t_0^{-1}t_1^{-2}-5t_0^{-2}t_1^{-1}\\
&\quad -2t_0^{-3}-2t_1^{-3}+t_0^{-1}t_1^{-3}+t_0^{-3}t_1^{-1}+69,\\
&LG(K_{2,1};t_0,t_1) = LG(K_{1,2};t_0,t_1)\\
&= -t_0^{-1}t_1^{-1}LG(K_{2,-1};t_0^{-1},t_1^{-1})+(1+t_0^{-1}t_1^{-1})LG(K_{0,-1};t_0^{-1},t_1^{-1}),\\
&LG(K_{2,2};t_0,t_1)\\
&= 4t_0^4t_1^4-9t_0^4t_1^3-9t_0^3t_1^4+6t_0^4t_1^2+30t_0^3t_1^3+6t_0^2t_1^4-t_0^4t_1-34t_0^3t_1^2-34t_0^2t_1^3\\
&\quad -t_0t_1^4+16t_0^3t_1+60t_0^2t_1^2+16t_0t_1^3-3t_0^3-45t_0^2t_1-45t_0t_1^2-3t_1^3+17t_0^2\\
&\quad +54t_0t_1+17t_1^2-5t_0^2t_1^{-1}-37t_0-5t_0^{-1}t_1^2-37t_1+t_0^2t_1^{-2}+16t_0t_1^{-1}\\
&\quad +16t_0^{-1}t_1+t_0^{-2}t_1^2-3t_0t_1^{-2}-18t_0^{-1}-3t_0^{-2}t_1-18t_1^{-1}+8t_0^{-1}t_1^{-1}+3t_0^{-2}\\
&\quad +3t_1^{-2}-t_0^{-1}t_1^{-2}-t_0^{-2}t_1^{-1}+39.
\end{aligned}$$

Proof. By Corollary 3, we have the following skein relation.

$LG\Big($  $\Big\}\, n$ full twists$\Big) = a(n)LG\Big($  $\Big) + (1-a(n))\,LG\Big($  $\Big)$

$+\dfrac{2(t_0-1)(t_1-1)(n-a(n))}{t_0t_1-1}LG\Big($  $\Big).$

Since $K_{n,\infty}$ $(=K_{\infty,n})$ is a split link, we have

$$LG(K_{n,\infty};t_0,t_1)=LG(K_{\infty,n};t_0,t_1)=0.$$

Applying the skein relation to $K_{2i-\epsilon,2j-\delta}$ twice, we have

$$\begin{aligned}
&LG(K_{2i-\epsilon,2j-\delta};t_0,t_1)\\
&=a(i)a(j)LG(K_{2-\epsilon,2-\delta};t_0,t_1)+a(i)(1-a(j))LG(K_{2-\epsilon,-\delta};t_0,t_1)
\end{aligned}$$

$$+(1-a(i))a(j)LG(K_{-\epsilon,2-\delta};t_0,t_1)+(1-a(i))(1-a(j))LG(K_{-\epsilon,-\delta};t_0,t_1).$$

By this equality, we have

$$\begin{aligned}
&LG(K_{2,1};t_0,t_1)\\
&= LG(K_{-2,-1};t_0^{-1},t_1^{-1})\\
&= a(-1)a(0)LG(K_{2,1};t_0^{-1},t_1^{-1})\\
&\quad + a(-1)(1-a(0))LG(K_{2,-1};t_0^{-1},t_1^{-1})\\
&\quad + (1-a(-1))a(0)LG(K_{0,1};t_0^{-1},t_1^{-1})\\
&\quad + (1-a(-1))(1-a(0))LG(K_{0,-1};t_0^{-1},t_1^{-1})\\
&= -t_0^{-1}t_1^{-1}LG(K_{2,-1};t_0^{-1},t_1^{-1})+(1+t_0^{-1}t_1^{-1})LG(K_{0,-1};t_0^{-1},t_1^{-1}),
\end{aligned}$$

where the first equality follows from Proposition 4 and the following equalities which are obtained by [8, Proposition 4.1].

$$K_{n,m}=K_{m,n}={K_{-n,-m}}^*={K_{-m,-n}}^*.$$

Furthermore, we have

$$\begin{aligned}
&LG(K_{n,m};t_0,t_1)=LG(K_{m,n};t_0,t_1)\\
&= LG(K_{-n,-m};t_0^{-1},t_1^{-1})=LG(K_{-m,-n};t_0^{-1},t_1^{-1}).
\end{aligned}$$

We obtain

$$\begin{gathered}
LG(K_{0,0};t_0,t_1),\ LG(K_{0,1};t_0,t_1),\ LG(K_{0,2};t_0,t_1),\\
LG(K_{1,1};t_0,t_1),\ LG(K_{1,-1};t_0,t_1),\ LG(K_{2,-1};t_0,t_1),
\end{gathered}$$

through the table of values of the invariant for all prime knots of up to 10 crossings [2], because the following facts are obtained by [8].

$$\begin{gathered}
K_{0,0}=4_1\#4_1,\ K_{0,1}=8_8,\ K_{0,2}={10_{137}}^*,\\
K_{1,1}={10_{155}}^*,\ K_{1,-1}=8_9,\ K_{2,-1}={10_{129}}^*.
\end{gathered}$$

It is easily checked that $K_{2,2}=(\sigma_2\sigma_4\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_4\sigma_1\sigma_3\sigma_2^{-1}\sigma_4^{-1}\sigma_1\sigma_3)^\wedge$, where $\sigma_1,\dots,\sigma_4$ are the standard generators of the 5-string braid group and $\wedge$ denotes the closure. Then we obtain $LG(K_{2,2};t_0,t_1)$ by computer-aided calculation from this braid representation of $K_{2,2}$ (cf. [2]). □

Lemma 6. *For $n,m\in\mathbb{Z}$,*

$$\begin{aligned}
&LG(K_{n,m};t^{1/2},-t^{-1/2})\\
&= \begin{cases}(t-3+t^{-1})^2 & \text{if } 2\mid n \text{ and } 2\mid m,\\ -t^3+3t^2-5t+7-5t^{-1}+3t^{-2}-t^{-3} & \text{if } 2\nmid n \text{ and } 2\nmid m,\\ 2t^2-6t+9-6t^{-1}+2t^{-2} & \text{otherwise.}\end{cases}
\end{aligned}$$

Proof. For $-1 \le n, m \le 2$, the result is checked by using Theorem 5. For $i, j \in \mathbb{Z}$, $\epsilon, \delta = 0, 1$, we have

$$\begin{aligned}
&LG(K_{2i-\epsilon,2j-\delta}; t^{1/2}, -t^{-1/2}) \\
&= (1-(-1)^i)(1-(-1)^j)2^{-2}LG(K_{2-\epsilon,2-\delta}; t^{1/2}, -t^{-1/2}) \\
&\quad + (1-(-1)^i)(1+(-1)^j)2^{-2}LG(K_{2-\epsilon,-\delta}; t^{1/2}, -t^{-1/2}) \\
&\quad + (1+(-1)^i)(1-(-1)^j)2^{-2}LG(K_{-\epsilon,2-\delta}; t^{1/2}, -t^{-1/2}) \\
&\quad + (1+(-1)^i)(1+(-1)^j)2^{-2}LG(K_{-\epsilon,-\delta}; t^{1/2}, -t^{-1/2}) \\
&= \begin{cases} (t-3+t^{-1})^2 & \text{if } 2 \mid 2i-\epsilon \text{ and } 2 \mid 2j-\delta, \\ -t^3+3t^2-5t+7-5t^{-1}+3t^{-2}-t^{-3} & \text{if } 2 \nmid 2i-\epsilon \text{ and } 2 \nmid 2j-\delta, \\ 2t^2-6t+9-6t^{-1}+2t^{-2} & \text{otherwise,} \end{cases}
\end{aligned}$$

where the first equality follows from Theorem 5 and the second equality follows from the results for $-1 \le n, m \le 2$. □

Lemma 7. *For $n, m \in \mathbb{Z}$,*

$$\begin{aligned}
LG(K_{n,m}; 2, 2) &= -15(-2)^{n+m-1} - 5(-2)^{n-2} - 5(-2)^{m-2} + 5(-2)^{-2}, \\
LG(K_{n,m}; -2, -2) &= 315 \cdot 2^{n+m+1} + 405 \cdot 2^{n-2} + 405 \cdot 2^{m-2} + 895 \cdot 2^{-2}.
\end{aligned}$$

Proof. For $-1 \le n, m \le 2$, the result is checked by using Theorem 5. For $i, j \in \mathbb{Z}$, $\epsilon, \delta = 0, 1$, we have

$$\begin{aligned}
&LG(K_{2i-\epsilon,2j-\delta}; 2, 2) \\
&= (2^{2i}-1)(2^{2j}-1)3^{-2}LG(K_{2-\epsilon,2-\delta}; 2, 2) \\
&\quad + (2^{2i}-1)(4-2^{2j})3^{-2}LG(K_{2-\epsilon,-\delta}; 2, 2) \\
&\quad + (4-2^{2i})(2^{2j}-1)3^{-2}LG(K_{-\epsilon,2-\delta}; 2, 2) \\
&\quad + (4-2^{2i})(4-2^{2j})3^{-2}LG(K_{-\epsilon,-\delta}; 2, 2) \\
&= ((-2)^{2i}-1)((-2)^{2j}-1)3^{-2} \\
&\quad \cdot (-15(-2)^{3-\epsilon-\delta} - 5(-2)^{-\epsilon} - 5(-2)^{-\delta} + 5(-2)^{-2}) \\
&\quad + ((-2)^{2i}-1)(4-(-2)^{2j})3^{-2} \\
&\quad \cdot (-15(-2)^{1-\epsilon-\delta} - 5(-2)^{-\epsilon} - 5(-2)^{-\delta-2} + 5(-2)^{-2}) \\
&\quad + (4-(-2)^{2i})((-2)^{2j}-1)3^{-2} \\
&\quad \cdot (-15(-2)^{1-\epsilon-\delta} - 5(-2)^{-\epsilon-2} - 5(-2)^{-\delta} + 5(-2)^{-2}) \\
&\quad + (4-(-2)^{2i})(4-(-2)^{2j})3^{-2} \\
&\quad \cdot (-15(-2)^{-1-\epsilon-\delta} - 5(-2)^{-\epsilon-2} - 5(-2)^{-\delta-2} + 5(-2)^{-2}) \\
&= -15(-2)^{2i-\epsilon+2j-\delta-1} - 5(-2)^{2i-\epsilon-2} - 5(-2)^{2j-\delta-2} + 5(-2)^{-2},
\end{aligned}$$

where the first equality follows from Theorem 5 and the second equality follows from the results for $-1 \le n, m \le 2$. In the same way, we have

$$LG(K_{n,m}; -2, -2) = 315 \cdot 2^{n+m+1} + 405 \cdot 2^{n-2} + 405 \cdot 2^{m-2} + 895 \cdot 2^{-2}.$$

□

Theorem 8 ([5]). *The LG invariant is complete for the Kanenobu knots.*

Proof. By Lemma 6, it is sufficient to show that

(1) if $LG(K_{2i,2j};t_0,t_1) = LG(K_{2i',2j'};t_0,t_1)$ then $\{i,j\} = \{i',j'\}$,
(2) if $LG(K_{2i,2j-1};t_0,t_1) = LG(K_{2i',2j'-1};t_0,t_1)$ then $(i,j) = (i',j')$,
(3) if $LG(K_{2i-1,2j-1};t_0,t_1) = LG(K_{2i'-1,2j'-1};t_0,t_1)$ then $\{i,j\} = \{i',j'\}$.

Let us suppose that $LG(K_{2i,2j};t_0,t_1) = LG(K_{2i',2j'};t_0,t_1)$. By Lemma 7, we have the following equalities.

$$\begin{aligned} 6\cdot 4^{i+j} - 4^i - 4^j &= 6\cdot 4^{i'+j'} - 4^{i'} - 4^{j'}, \\ 56\cdot 4^{i+j} + 9\cdot 4^i + 9\cdot 4^j &= 56\cdot 4^{i'+j'} + 9\cdot 4^{i'} + 9\cdot 4^{j'}. \end{aligned}$$

These equalities imply that $4^{i+j} = 4^{i'+j'}$ and $4^i + 4^j = 4^{i'} + 4^{j'}$. Then we have $\{i,j\} = \{i',j'\}$.

Let us suppose that $LG(K_{2i,2j-1};t_0,t_1) = LG(K_{2i',2j'-1};t_0,t_1)$. By Lemma 7, we have the following equalities.

$$\begin{aligned} 6\cdot 4^{i+j} + 2\cdot 4^i - 4^j &= 6\cdot 4^{i'+j'} + 2\cdot 4^{i'} - 4^{j'}, \\ 56\cdot 4^{i+j} + 18\cdot 4^i + 9\cdot 4^j &= 56\cdot 4^{i'+j'} + 18\cdot 4^{i'} + 9\cdot 4^{j'}. \end{aligned}$$

These equalities imply that $4^{i+j} + 9\cdot 4^j = 4^{i'+j'} + 9\cdot 4^{j'}$. Let f be the map $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Q}$ defined by $f(i,j) = 4^{i+j} + 9\cdot 4^j$. Then it is sufficient to show that the map f is injective.

To prove this, we note that every element of $f(\mathbb{Z}\times\mathbb{Z})$ is expressed as a product $p\cdot 2^k$ ($p, k \in \mathbb{Z}$, $2 \nmid p$) in a unique way. By the definition, we have

$$f(i,j) = \begin{cases} (4^i + 9)\cdot 2^{2j} & \text{if } i > 0, \\ 5\cdot 2^{2j+1} & \text{if } i = 0, \\ (1 + 9\cdot 4^{-i})\cdot 2^{2i+2j} & \text{if } i < 0. \end{cases}$$

Since $4^{i_1} + 8$ is never expressed as a product $9\cdot 2^{2i_3}$ for any $i_1, i_3 > 0$, we have $4^{i_1} + 9 \neq 1 + 9\cdot 4^{i_3}$. If $f(i,j) = f(i',j')$, then either $ii' > 0$ or $i = i' = 0$, which follows from $4^{i_1} + 9 \neq 1 + 9\cdot 4^{i_3}$ and $2^{2j_2+1} \neq 2^{2j_1}, 2^{-2i_3+2j_3}$. By the uniqueness of the expression, f is injective on $\mathbb{Z}_{>0}\times\mathbb{Z}$, $\{0\}\times\mathbb{Z}$, $\mathbb{Z}_{<0}\times\mathbb{Z}$. Thus the map f is injective.

Let us suppose that $LG(K_{2i-1,2j-1};t_0,t_1) = LG(K_{2i'-1,2j'-1};t_0,t_1)$. By Lemma 7, we have the following equalities.

$$\begin{aligned} 3\cdot 4^{i+j} + 4^i + 4^j &= 3\cdot 4^{i'+j'} + 4^{i'} + 4^{j'}, \\ 28\cdot 4^{i+j} + 9\cdot 4^i + 9\cdot 4^j &= 28\cdot 4^{i'+j'} + 9\cdot 4^{i'} + 9\cdot 4^{j'}. \end{aligned}$$

These equalities imply that $4^{i+j} = 4^{i'+j'}$ and $4^i + 4^j = 4^{i'} + 4^{j'}$. Then we have $\{i,j\} = \{i',j'\}$. □

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Osaka University