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DIHEDRAL COCYCLE INVARIANTS OF TWIST-SPUN 2-BRIDGE KNOTS

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1. Introduction

A *surface knot* is an oriented connected closed surface F embedded in 4-space $\mathbf{R}^4$, and a surface knot F is a 2-knot if F is homeomorphic to the 2-sphere. Zeeman [19] gave a method to construct a 2-knot from a classical knot K for any integer r. It is called the *r-twist-spin* of K, or the *r-twist spun* K, and is denoted by $\tau^r K$. In [2], J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito introduced the quandle cocycle invariants of 2-knots, and calculated the cocycle invariant of the dihedral quandle of order 3. S. Asami and S. Satoh [1] calculated the cocycle invariants of r-twist-spun torus knots $\tau^r T(m,n)$ associated with 3-cocycles of the dihedral quandles of order p by observing motion pictures for any odd prime integer p.

Let α and β be relatively odd integers such that α is positive and $-\alpha < \beta < \alpha$. Then the 2-bridge knot $S(\alpha,\beta)$ is defined (cf. [18]). In this paper, we calculate the cocycle invariants of r-twist-spun 2-bridge knots $\tau^r S(\alpha,\beta)$. The dihedral cocycle invariant of 2-knots Φ_{θ_p} is defined in §2 and the invariants of knots $\Psi^*_{\theta_p}$ are defined in §3. These three invariants are valued in the quotient of the Laurent polynomial ring $\mathbf{Z}[t^{\pm 1}]/(t^p-1)$. We have the following theorem.

Theorem 1.1. *Let n be an odd integer in $\{1,3,\cdots,2\alpha-1\}$ such that $n\beta = 1 \pmod{2\alpha}$ and let p be an odd prime integer.*

(i) *For the 2-bridge knot $S(\alpha,\beta)$,*

$$\Psi^*_{\theta_p}(S(\alpha,\beta)) = \begin{cases} p\sum_{s=0}^{p-1} t^{-\frac{n\alpha}{p}s^2}, & \textit{if } \alpha \textit{ is divisible by } p\textit{:} \\ p, & \textit{otherwise.} \end{cases}$$

(ii) *For the r-twist-spin of 2-bridge knot $\tau^r S(\alpha,\beta)$,*

$$\Phi_{\theta_p}(\tau^r S(\alpha,\beta)) = \begin{cases} p\sum_{s=0}^{p-1} t^{-\frac{n\alpha r}{p}s^2}, & \textit{if } \alpha \textit{ is divisible by } p \textit{ and } r \textit{ is an even:} \\ p, & \textit{otherwise.} \end{cases}$$

For a 2-knot F, the surface that is obtained from F by attaching trivial 1-handles is called by a *stabilized surface* of F, and stabilized surface with genus g are denoted by $S_g F$. For a surface knot F, the surface knot obtained by reversing the orientation of F is denoted by $-K$ and called the reverse of F. A surface knot F is *non-invertible* if F is not ambiently isotopic to $-F$. The non-invertibility of some 2-knots was studied by the Alexander modules, the Farber-Levine invariants, the Casson-Gordon invariants, and so on (cf. [4, 6, 7, 8, 12, 16]). By using rack or quandle cocycle invariants, it is proved that a twist-spun trefoil knot and its stabilized surfaces are non-invertible [2, 15, 17], and some twist-spun torus knots and their stabilized surfaces are non-invertible [1]. And in [15], a trefoil knot is not ambiently isotopic to its mirror image.

Corollary 1.2. *Let r be an even integer. Then $S_g\tau^r S(\alpha,\beta)$ and $-S_g\tau^r S(\alpha,\beta)$ can be distinguished by the dihedral quandle cocycle invariant Φ_{θ_p} (and hence $S_g\tau^r S(\alpha,\beta)$ is non-invertible) if there is a positive integer p satisfying the following conditions:*

- (i) *p is an odd prime with $p \equiv 3 \pmod 4$.*
- (ii) *α is divisible by p and indivisible by p^2.*
- (iii) *r is indivisible by p.*

For example, $p = 3$ satisfies the above conditions when $(\alpha,\beta) = (3,1)$ and $r = 2$. Thus the 2-twist-spun trefoil knot $\tau^2 S(3,1)$ is non-invertible.

Schubert [18] proved that a 2-bridge knot $S(\alpha,\beta)$ is not ambiently isotopic to its mirror-image if and only if $\beta^2 \not\equiv -1 \pmod \alpha$. Recently, C. McA. Gordon [7] proved that if $r \geq 2$ then the r-twist-spun 2-bridge knot K is non-invertible if and only if K is not ambiently isotopic to its mirror-image. Therefore, for $r \geq 2$, $\tau^r S(\alpha,\beta)$ is non-invertible if and only if $\beta^2 \not\equiv -1 \pmod \alpha$. By Theorem 1.1, $\Phi_{\theta_p}(\tau^r S(\frac{\beta^2+1}{2},\beta)) = p$ for odd r. Hence there are infinite many non-invertible twist-spun 2-bridge knots such that they cannot be distinguished from their reverses by any dihedral quandle cocycle invariant.

We review the definition of quandle cocycle invariants in §2, the definition of shadow colorings of tangle diagrams in §3, and Asami and Satoh's results in §4. In §5, we calculate the invariants of 2-bridge knots and twist-spun 2-bridge knots and prove Theorem 1.1 and Corollary 1.2.

2. Cocycle invariants

A *quandle* (cf. [5, 9, 10, 13]) is a set Q with a binary operation $* : Q \times Q \longrightarrow Q$ satisfying the following properties:

- (Q1) For any $q \in Q$, $q * q = q$.
- (Q2) For any $q_1, q_2 \in Q$, there is a unique $q_3 \in Q$ such that $q_1 = q_3 * q_2$.
- (Q3) For any $q_1, q_2, q_3 \in Q$, $(q_1 * q_2) * q_3 = (q_1 * q_3) * (q_2 * q_3)$

The homology and cohomology for quandles are developed in [2]. For an abelian group G, let $C^n(Q;G)$ be the free abelian group generated by the maps $f : Q^n \longrightarrow G$ such that $f(q_1,\cdots,q_n) = 0$ if $q_i = q_{i+1}$ for some $i \in \{1,\cdots,n\}$. The coboundary map $\delta^n : C^n(Q;G) \longrightarrow C^{n+1}(Q;G)$ is given by

$$\begin{aligned}(\delta^n f)(q_1,\cdots,q_{n+1}) \quad = \quad & \sum_{k=2}^{n+1}(-1)^k\{f(q_1,\cdots,q_{k-1},q_{k+1},\cdots,q_{n+1}) \\ & -f(q_1 * q_k,\cdots,q_{k-1} * q_k, q_{k+1},\cdots,q_{n+1})\}.\end{aligned}$$

The quandle cohomology group $H^*(Q;G)$ is defined by $C^*(Q;G) = \{C^*(Q;G), \delta^*\}$ in an usual manner, and the cocycle and coboundary groups are defined by $Z^*(Q;G)$ and $B^*(Q;G)$ respectively.

Example 2.1. The set $\{0, 1, \cdots, p-1\}$ becomes a quandle under the binary operation $a * b = 2b - a \pmod p$, which is called the *dihedral quandle* of order p and denoted by R_p. Mochizuki [14] proved that $H^3(R_p; \mathbf{Z}_p) \cong \mathbf{Z}_p$ for any odd prime p, and gave its generator. S. Asami and S. Satoh [1] changed the Mochizuki's 3-cocycle more simply as follows:

$$\theta_p(q_1, q_2, q_3) = (q_1 - q_2)\frac{(2q_3 - q_2)^p + q_2^p - 2q_3^p}{p} \in \mathbf{Z}/p\mathbf{Z}(\cong \mathbf{Z}_p),$$

for $(q_1, q_2, q_3) \in Q^3$. $[\theta_p] \in H^3(R_p; \mathbf{Z}_p)$ is also a generator. We will use this θ_p in §3, §4 and §5.

For a surface knot F in $\mathbf{R}^4$, modifying it slightly, we assume that the projection $p : F \longrightarrow \mathbf{R}^3$ to the 3-space is a generic map. The singularity of the projection consists of double point curves and isolated triple points and isolated branch points. Removing a small regular neighborhood of the under-curve of the double curve, we have a compact surface in $\mathbf{R}^3$. We call it a *diagram* for the surface knot F. Let D be the diagram of F, and $\Sigma(D)$ the set of the connected regions of D. Using the orientation of F and $\mathbf{R}^3$, we give an orientation normal of each connected region of D. A map $C : \Sigma(D) \longrightarrow Q$ into a quandle Q is a *Q-coloring* of D if it satisfies the following condition along every double point curve; if λ_1 and λ_2 are under-sheets and μ is the over-sheet, where the orientation normal of μ points from λ_1 to λ_2, then $C(\lambda_1) * C(\mu) = C(\lambda_2)$ (Fig.1). We denote the set of all Q-colorings of D by $Col_Q(D)$.

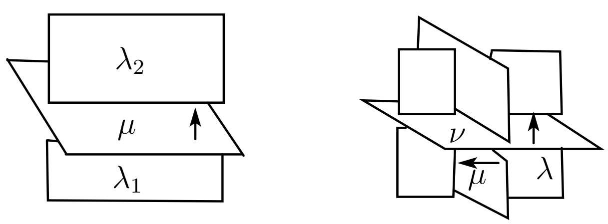

FIGURE 1

Let Q be a finite quandle and $f \in Z^3(Q;G)$ a 3-cocycle. We define the *Boltzmann weight* at a triple point η by

$$W_f(\eta; C) = f(C(\lambda), C(\mu), C(\nu))^{\epsilon(\eta)} \in G$$

where λ, μ and ν are the sheets around η such that ν is the top sheet, μ is the middle sheet from which the orientation normal of ν points, and λ is the bottom sheet from which the orientation normals of μ and ν point, and $\epsilon(\eta)$ is the sign of the triple point η (Fig.1). Consider the state-sum

$$\Phi_f(F) = \sum_{C \in Col_Q(D)} \prod_{\eta} W_f(\eta; C) \in \mathbf{Z}[G]$$

valued in the group ring $\mathbf{Z}[G]$, where η runs all triple points of D.

Proposition 2.2 ([2]). *The state-sum $\Phi_f(F)$ does not depend on the choice of a diagram D of F.*

The state-sum $\Phi_f(F)$ is called the *quandle cocycle invariant* of F associated with f. If θ_p is a 3-cocycle of dihedral quandle R_p given in as Example 2.1, we call Φ_{θ_p} by a *dihedral cocycle invariant*.

3. Shadow colorings of tangles

Let D_T be a connected arc with boundary on a 2-disk B, where $\partial D_T = D_T \cap \partial B$, and D_K be a knot diagram on the 2-sphere S^2. Fix a point $e \in D_K$. Let $N(e)$ be a neighborhood of e such that $(N(e), N(e) \cap D_K)$ is homeomorphic to a standard ball pair (B^2, B^1). A *tangle diagram given by removing the neighborhood of e from a diagram D_K on the 2-sphere S^2* is the arc D_T on the 2-disk B such that (B, D_T) is homeomorphic to $(S^2 - N(e), D_K - D_K \cap N(e))$.

Let T be a one-string tangle diagram on a 2-disk B of a classical knot K, and $\Sigma(T)$ the set of arcs of T. A map $C : \Sigma(T) \longrightarrow Q$ into a quandle Q is a *Q-coloring* of T if it satisfies the following condition at each crossing x; if λ_1 and λ_2 are under-arcs separated by an over-arc μ, where λ_1 is the right side of μ, then $C(\lambda_1)*C(\mu) = C(\lambda_2)$. The pair $(C(\lambda_1), C(\mu)) \in Q^2$ is called the *quandle pair* at x. A *shadow Q-coloring of T extending C* is a map $C^* : \Sigma^*(T) \longrightarrow Q$, where $\Sigma^*(T)$ is the union of $\Sigma(T)$ and the set of regions of the 2-disk B separated by the underlying immersed curve of T, satisfying the following conditions: (i) C^* restricted to $\Sigma(T)$ coincides with C, (ii) if R_1 and R_2 are regions separated by an arc λ, where R_1 is on the right-side of λ, then $C^*(R_1) * C^*(\lambda) = C^*(R_2)$, and (iii) $C^*(\lambda_+) = C^*(R_0)$, where λ_+ is the initial arc of T and R_0 is the right side region of λ_+. The set of shadow Q-colorings of T is denoted by $Col^*_Q(T)$. Note that C^* always exists uniquely for a given C (cf. [3]). Let $f \in Z^3(Q; G)$ be a 3-cocycle of a finite quandle Q. Define the Boltzmann weight at each crossing x by $W^*_f(x; C^*) = f(s, q_1, q_2)^{\epsilon(x)} \in G$, where (q_1, q_2) is the quandle pair at x and s is the color of the specified region as shown in Fig.2. And $(s, q_1, q_2) \in Q^3$ is called a *quandle triple* at x. Let $X_2(T)$ be the set of crossing points of T. Consider the state-sum

$$\Psi^*_f(T) = \sum_{C^* \in Col^*_Q(T)} \prod_{x \in X_2(T)} W^*_f(x; C^*) \in \mathbf{Z}[G],$$

valued in group ring $\mathbf{Z}[G]$ (cf. [1, 2, 15]).

Proposition 3.1 ([1, 2, 15]). *If T and T' are one-string tangle diagrams of a classical knot K, then $\Psi^*_f(T) = \Psi^*_f(T')$, so that we will denote $\Psi^*_f(T)$ by $\Psi^*_f(K)$.*

Let R_p be the dihedral quandle of order p, where p is an odd prime. Also we refer to (shadow) R_p-coloring of the tangle diagram T as (*shadow*) *p-coloring* of T. A (shadow) p-coloring of T is *non-trivial* if the image of the coloring consists of at least two elements. Let $\theta_p \in Z^3(R_p; \mathbf{Z}_p)$ be the 3-cocycle given in Example 2.1. Taking the coefficient group $\mathbf{Z}_p = \langle t | t^p = 1 \rangle$, we identify the group ring $\mathbf{Z}[\mathbf{Z}_p]$ with the quotient of the Laurent polynomial ring $\mathbf{Z}[t^{\pm 1}]/(t^p - 1)$. Let $\rho^r : \mathbf{Z}[t^{\pm 1}]/(t^p - 1) \longrightarrow \mathbf{Z}[t^{\pm 1}]/(t^p - 1)$ be the map sending by $t \longrightarrow t^r$. By S. Asami and S. Satoh, the

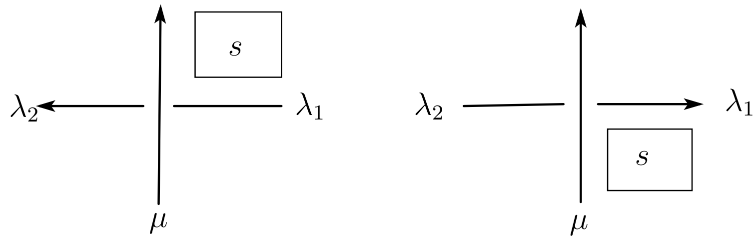

FIGURE 2

dihedral quandle cocycle invariant of r-twist spanned knot $\tau^r K$ is translated in terms of a tangle diagram of K.

Proposition 3.2 ([1]). (i) *If r is odd, then we have $\Phi_{\theta p}(\tau^r K) = p$.*
(ii) *If r is even, then we have $\Phi_{\theta p}(\tau^r K) = \rho^r \Psi^*_{\theta_p}(K)$.*

Lemma 3.3. *Let C^* be a shadow p-coloring of a tangle diagram T for an odd prime p. At any crossing point x, let λ be the under-arc oriented inward at x, R the right side region of λ, and μ the over-arc (Fig.3). Then*

$$W^*_{\theta_p}(x; C^*) = \theta_p(C^*(R), C^*(\lambda), C^*(\mu)).$$

Proof. Since $(q_1 * q_2) * q_2 = q_1$ for any $q_1, q_2 \in R_p$, $q_1 * q_2 = q_1 *^{-1} q_2$. If x is a positive crossing, then the quandle triple at x is $(C^*(R), C^*(\lambda), C^*(\mu))$, if x is a negative crossing, then the quandle triple at x is $(C^*(R) *^{-1} C^*(\mu), C^*(\lambda) *^{-1} C^*(\mu), C^*(\mu)) = (C^*(R) * C^*(\mu), C^*(\lambda) * C^*(\mu), C^*(\mu))$. Since θ_p satisfies $\theta_p(q_1, q_2, q_3) = \theta_p(q_1 * q_3, q_2 * q_3, q_3)^{-1}$ for any $q_1, q_2, q_3 \in R_p$, $\theta_p(C^*(R), C^*(\lambda), C^*(\mu)) = \theta_p(C^*(R) * C^*(\mu), C^*(\lambda) * C^*(\mu), C^*(\mu))^{-1}$. Therefore $W^*_{\theta_p}(x; C^*) = \theta_p(C^*(R), C^*(\lambda), C^*(\mu))$. □

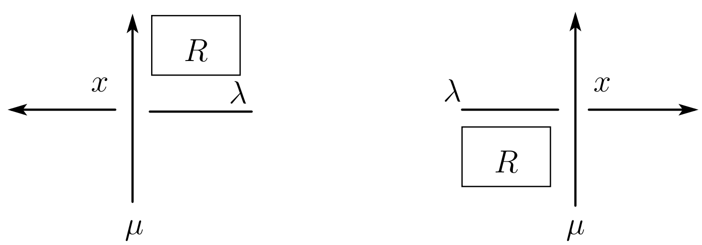

FIGURE 3

We call the triple $(C^*(R), C^*(\lambda), C^*(\mu)) \in Q^3$ the *i-triple* at x of the tangle diagram and denote it by $I_{C^*}(x)$, where C^*, R, λ, μ are as in Lemma 3.3.

4. CONCRETE CALCULATIONS OF TWIST SPANNED 2-BRIDGE KNOTS

Let α and β are relatively prime odd integers such that α is an positive integer and $-\alpha < \beta < \alpha$. Then there is a diagram $D_{S(\alpha,\beta)}$ on the 2-sphere of the 2-bridge knot presented in Schubert's normal form (cf. [18]). This is a regular projection with two upper-bridges such that each of the two lower-bridges goes alternately under them. Each upper-bridge contains the same number, say $\alpha - 1$, of crossing points (so that the total number of crossings is $2\alpha-2$). These are labeled $1, 2, \cdots, \alpha-1$, in the order opposite to that induced by a selected orientation, and each index is prefixed by $+$ or $-$ according as the crossing is right- or left-handed. Then, as one travels along an lower-bridge in the direction indicated by the orientation, the upper-bridges are gone under alternately, and the signed indices of these $\alpha-1$ crossings, in the order in which they are encountered belong to the residue classes $\beta, 2\beta, \cdots, (\alpha-1)\beta$ (mod 2α). For example, $S(5,3)$ is depicted in Fig.4.

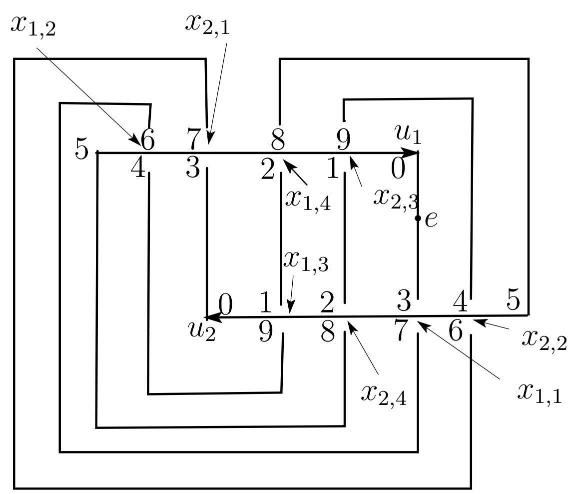

FIGURE 4. $S(5,3)$

Let u_1 and u_2 be the upper-bridges, v_1 and v_2 be the lower-bridges, where the orientation of v_1 points from u_1 to u_2, and $x_{i,1}, x_{i,2}, \cdots, x_{i,\alpha-1}$ be the crossing points on v_i, where the orientation of v_i points from $x_{i,j}$ to $x_{i,j+1}$. Fix a base point e on the under-arc oriented inward at $x_{1,1}$. See Fig.6. Let $T(\alpha,\beta)$ be a tangle diagram given by removing the neighborhood $N(e)$ of e from $D_{S(\alpha,\beta)}$. We will use the same symbol $u_i, x_{i,j}$ on $T(\alpha,\beta)$. The *characteristic region* of $T(\alpha,\beta)$ is the region of the right side of a segment between crossing points on u_2 with label 0 and 1. See Fig.7.

Let a_+ and a_- be the color of a initial arc and terminal arc of $T(\alpha,\beta)$, respectively. By the definition of the shadow coloring of the tangle diagram, $a_+ * a_- = a_+$. Since p is an odd integer, a_- is a unique color. Hence $a_- = a_+$. We consider the shadow p-coloring, where the colors of u_1 and u_2 are a and b, respectively. If there is this shadow p-coloring, we denote it by $C^*(a,b)$. For $C^*(a,b)$, let $\sigma^*_{a,b}$ be the color of the characteristic region.

Lemma 4.1. *For* $C^*(a,b)$,

(i) $I_{C^*(a,b)}(x_{1,j}) = (ja-(j-1)b, ja-(j-1)b, b)$ *for odd* j.
(ii) $I_{C^*(a,b)}(x_{1,j}) = (jb-(j-1)a, jb-(j-1)a, a)$ *for even* j.
(iii) $I_{C^*(a,b)}(x_{2,j}) = (ja-(j-2)b-\sigma^*_{a,b}, ja-(j-1)b, b)$ *for even* j.
(iv) $I_{C^*(a,b)}(x_{2,j}) = ((j-1)(b-a)+\sigma^*_{a,b}, jb-(j-1)a, a)$ *for odd* j.

Proof. If $I_{C^*(a,b)}(x_{i,j}) = (q_1, q_2, q_3)$, then the color of the under-arc λ oriented outward at $x_{i,j}$ is $q_2 * q_3$ and the color of the right side region of λ is $q_1 * q_3$ since $(q*q')*q' = q$ for any $q, q' \in R_p$. Because λ is the under-arc oriented inward at $x_{i,j+1}$, $I_{C^*(a,b)}(x_{i,j+1}) =$

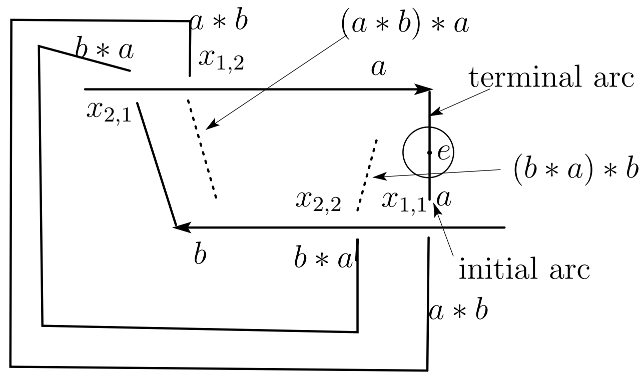

FIGURE 5

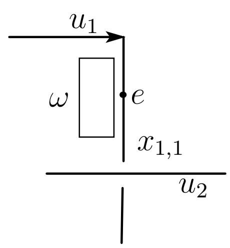

FIGURE 6

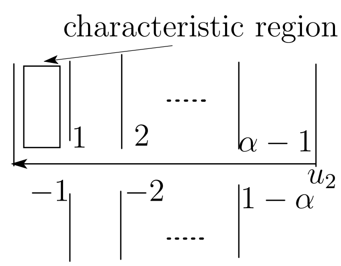

FIGURE 7

$(q_1 * q_3, q_2 * q_3, q_4)$, where the color of the over-arc at $x_{i,j+1}$ is q_4. In $T(\alpha,\beta)$, any lower-bridge crosses two upper-bridges alternately. By the definition of $C^*(a,b)$, $I_{C^*(a,b)}(x_{1,1}) = (a,a,b)$. Therefore $I_{C^*(a,b)}(x_{1,2}) = (a*b, a*b, a)$, $I_{C^*(a,b)}(x_{1,3}) = ((a*b)*a, (a*b)*a, b)$. By induction, $I_{C^*(a,b)}(x_{1,j}) = ((a*b)*\cdots)*a, (a*b)\cdots)*a, b)$ for odd j, and $I_{C^*(a,b)}(x_{1,j}) = ((a*b)*\cdots)*b, (a*b)\cdots)*b, a)$ for even j. Since $i*j = 2j-i \pmod p$, we have the equations (i) and (ii). On the other hand, $I_{C^*(a,b)}(x_{2,1}) = (\sigma^*_{a,b}, b, a)$. By the similar argument, we have the equations (iii) and (iv). □

Let $C^*(a,b)$ be the shadow p-coloring of $T(\alpha,\beta)$ as above. By Lemma 4.1, the color of the under-arc λ_1 and λ_2 oriented outward at $x_{1,\alpha-1}$ and $x_{2,\alpha-1}$ are $\alpha a-(\alpha-1)b$ and $\alpha b-(\alpha-1)a$, respectively. Since the arc λ_1 and λ_2 connect with the upper-bridge u_2 and u_1, respectively, $\alpha(a-b)=0 \pmod p$. If α is indivisible by p, then $a=b$. We can check that the color of all arc is a if $a=b$, hence $T(\alpha,\beta)$ does not have any non-trivial shadow p-colorings. If α is divisible by p, then the equation $\alpha(a-b)=0 \pmod p$ holds for any $a,b\in R_p$. Therefore $T(\alpha,\beta)$ has non-trivial shadow p-colorings.

Proposition 4.2. *Let $C^*(a,b)$ and $\sigma^*_{a,b}$ be as in Lemma.4.1. If α is divisible by p for an odd prime p , then*

$$\sum_{x\in X_2(T(\alpha,\beta))} W^*_{\theta_p}(x; C^*(a,b)) = \frac{\alpha}{p}(\sigma^*_{a,b}-b)(b-a) \in \mathbf{Z_p}$$

Proof. Let $X_2^{i,o} = \{x_{i,j} \in X_2(T(\alpha,\beta)) |$ odd $j\}$ and $X_2^{i,e} = \{x_{i,j} \in X_2(T(\alpha,\beta)) |$ even $j\}$ be the sets for $i \in \{1,2\}$. For a crossing point in $X_2^{1,o}, X_2^{1,e}, X_2^{2,e}$, and $X_2^{2,o}$, the i-triple at x is given in (i), (ii), (iii), and (iv) of Lemma 4.1. Therefore by Lemma 3.3 and $\theta_p(q_1,q_1,q_2)=0$ for any q_1,q_2,

$$\sum_{x\in X_2^{1,o}} W^*_{\theta_p}(x;C^*(a,b)) = 0, \sum_{x\in X_2^{1,e}} W^*_{\theta_p}(x;C^*(a,b)) = 0,$$

$$\sum_{x\in X_2^{2,e}} W^*_{\theta_p}(x;C^*(a,b)) = \sum_{j=1}^{\frac{\alpha-1}{2}} \theta_p(2ja-(2j-2)b-\sigma^*_{a,b}, 2ja-(2j-1)b, b),$$

$$\sum_{x\in X_2^{2,o}} W^*_{\theta_p}(x;C^*(a,b)) = \sum_{j=1}^{\frac{\alpha-1}{2}} \theta_p((2j-2)(b-a)+\sigma^*_{a,b}, (2j-1)b-(2j-2)a, a).$$ Therefore we have

$$\sum_{x\in X_2(T(\alpha,\beta))} W^*_{\theta_p}(x;C^*(a,b)) = +\sum_{j=1}^{\frac{\alpha-1}{2}} \theta_p(2ja-(2j-2)b-\sigma^*_{a,b}, 2ja-(2j-1)b, b)$$
$$+\sum_{j=1}^{\frac{\alpha-1}{2}} \theta_p((2j-2)(b-a)+\sigma^*_{a,b}, (2j-1)b-(2j-2)a, a)$$
$$= (b-\sigma^*_{a,b})\sum_{j=1}^{\frac{\alpha-1}{2}} \frac{((2j+1)b-2ja)^p+(2ja-(2j-1)b)^p-2b^p}{p}$$
$$+(\sigma^*_{a,b}-b)\sum_{j=1}^{\frac{\alpha-1}{2}} \frac{((2ja-(2j-1)b)^p+((2j-1)b-(2j-2)a)^p-2a^p}{p}$$
$$= (\sigma^*_{a,b}-b)\frac{-(\alpha b-(\alpha-1)a)^p+b^p+(\alpha-1)(b^p-a^p)}{p}$$
Since α is divisible by p, $(\alpha b-(\alpha-1)a)^p = -(\alpha-1)^p a^p = a^p \pmod{p^2}$,

$$\begin{aligned} \sum_{x\in X_2(T(\alpha,\beta))} W^*_{\theta_p}(x;C^*(a,b)) &= (\sigma^*_{a,b}-b)\frac{-a^p+\alpha b^p-(\alpha-1)a^p}{p} \\ &= (\sigma^*_{a,b}-b)\frac{\alpha}{p}(b^p-a^p) \\ &= \frac{\alpha}{p}(\sigma^*_{a,b}-b)(b-a). \end{aligned}$$

□

Since 2α and β are relatively prime integers, there is a unique odd integer $n \in \{1,3,\cdots,2\alpha-1\}$ such that $n\beta = 1 \pmod{2\alpha}$.

Lemma 4.3. *Let n be an odd integer in $\{1,3,\cdots,2\alpha-1\}$ such that $n\beta = 1 \pmod{2\alpha}$. For any shadow p-coloring $C^*(a,b)$ of $T(\alpha,\beta)$, $\sigma^*_{a,b} = na-(n-1)b$.*

Proof. By definition of Schubert's normal form and that n is odd , $x_{1,n}$ is a positive crossing with label 1 on u_2 if $1 \le n \le \alpha-2$. Therefore $\sigma^*_{a,b}$ is the first element of

i-triple at $x_{1,n}$. By Lemma 4.1, $\sigma^*_{a,b} = (n-1)(a-b)+a$. By $(2\alpha-n)\beta = -n\beta = -1 \pmod{2\alpha}$, $x_{1,2\alpha-n}$ is a negative crossing with label 1 on u_2 if $\alpha \le n \le 2\alpha-1$. Then the characteristic region is a diagonal position to the right side region of the under-arc oriented inward at $x_{1,2\alpha-n}$. Therefore $\sigma^*_{a,b} = (((2\alpha-n-1)(a-b)+a) * ((2\alpha-n)a-(2\alpha-n-1)b)) * b = na-(n-1)b \pmod{p}$. □

Proof of Theorem.1.1. By Proposition 4.2 and Lemma 4.3, if α is divisible by p for odd prime p, $\sum_{x\in X_2(T(\alpha,\beta))} W^*_{\theta_p}(x; C^*(a,b)) = \frac{\alpha}{p}(\sigma^*_{a,b}-b)(b-a)$ $= \frac{\alpha}{p}((na-(n-1)b-b)(b-a) = \frac{\alpha}{p}n(a-b)(b-a)$. For any integer $k \in \{0,1,\cdots,p-1\}$, $\sharp\{C^*(a,b) \in Col^*_{R_p}(T(\alpha,\beta)) | a-b = k \pmod{p}\} = p$. If α is not divisible by p, $T(\alpha,\beta)$ does not have any non-trivial p-coloring. Therefore we have (i) of this theorem. By Proposition 3.1, we have (iii) of this theorem. □

Proof of Corollary.1.2. Since $\tau^{-r}K$ is an orientation reserved knot of $\tau^r K$ for any a classical knot K, it is sufficient to prove that $\Phi_{\theta_p}(\tau^r S(\alpha,\beta)) \ne \Phi_{\theta_p}(\tau^{-r} S(\alpha,\beta))$. This can be seen easily from the fact that $p(\sum_{s=0}^{p-1} t^{-Ns^2}) \ne p(\sum_{s=0}^{p-1} t^{Ns^2})$ in $\mathbf{Z}[t^{\pm1}]/(t^p-1)$ if and only if $p \equiv 3 \pmod{4}$ and N is indivisible by p. And n is indivisible by p if $n \in \{1,3,\cdots,2\alpha-1\}$ is an odd integer such that $n\beta = 1 \pmod{2\alpha}$. Since the quandle cocycle invariants are independent on attaching trivial 1-handles, therefore we have the theorem by Theorem 1.1. □

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