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TWO TUNNELS OF A TUNNEL NUMBER ONE LINK

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Abstract. We give a condition for a pair of unknotting tunnels of a non-trivial tunnel number one link to give a genus three Heegaard splitting of the link complement and show that every 2-bridge link has such a pair of unknotting tunnels.

1. Introduction

A *compression body* H is a 3-manifold obtained from a compact connected closed surface S by attaching 2-handles to $S \times I$ on $S \times \{1\}$ and capping off any resulting 2-sphere boundary components with 3-handles. $S \times \{0\}$ is denoted by $\partial_+ H$ and $\partial H - \partial_+ H$ is denoted by $\partial_- H$. A compression body H is called a *handlebody* if $\partial_- H = \emptyset$.

If a compact 3-manifold M is the union of two compression bodies H_1 and H_2 along their common "plus" boundary $S = \partial_+ H_1 = \partial_+ H_2$, we call the decomposition $M = H_1 \cup_S H_2$ a *Heegaard splitting* of M and S a *Heegaard surface* of M. The minimum number of the genus of S among all Heegaard splittings of M is called the *Heegaard genus* (or *genus*) of M.

Suppose $H_1 \cup_S H_2$ is a Heegaard splitting of a 3-manifold M and α is a properly embedded arc in H_2 parallel to an arc in S. That is, there is an embedded disk D in H_2 whose boundary is the union of α and an arc in $\partial_+ H_2$. Now add a neighborhood of α to H_1 and delete it from H_2. Once again the result is a Heegaard splitting $H_1' \cup_{S'} H_2'$, where the genus of each H_i' is one greater than H_i. This process is called a *stabilization* of S.

Every compact 3-manifold can be triangulated and any two triangulations of a 3-manifold are PL-equivalent ([1], [5]). It follows that every compact 3-manifold has a Heegaard splitting and any two Heegaard splittings of a 3-manifold have a common stabilization. In fact, there is no example of distinct Heegaard splittings of a same closed 3-manifold which cannot be made isotopic by a single stabilization of one of the splittings, and sufficient stabilizations of the other to ensure that the genus of the two surfaces is the same. This makes the following conjecture very optimistic.

Conjecture 1.1. [7] *Suppose $H_1 \cup_S H_2$ and $H_1' \cup_{S'} H_2'$ are Heegaard splittings of the same 3-manifold of, genus $g \leq g'$ respectively. Then the splittings obtained by one stabilization of S' and $g' - g + 1$ stabilizations of S are isotopic.*

A *tunnel system* (or *tunnels*) of a knot or a link K is a collection of disjoint embedded arcs $t_1, t_2, \cdots, t_n$ in S^3 with $K \cap \bigcup_{i=1}^{n} t_i = \bigcup_{i=1}^{n} \partial t_i$ such that $H = \overline{S^3 - N(K \cup \bigcup_{i=1}^{n} t_i)}$ is a genus $n+1$ handlebody. (Here $N(X)$ denotes a regular neighborhood of X.)

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The tunnel system gives rise to a Heegaard splitting of the exterior of K

$$\overline{S^3 - N(K)} = H \cup_{\partial H} \overline{N(K \cup \bigcup_{i=1}^{n} t_i) - N(K)}$$

where $N(K)$ is contained in the interior of $N(K \cup \bigcup_{i=1}^{n} t_i)$. The minimum of such number n is called the *tunnel number* of K. If the tunnel number of K is 1, the tunnel is called an *unknotting tunnel* of K.

For a tunnel number one knot K, we consider two non-isotopic unknotting tunnels t_1, t_2 and corresponding Heegaard surfaces S_1, S_2 of the exterior of K. Now suppose $V = \overline{S^3 - N(K \cup t_1 \cup t_2)}$ is a genus three handlebody. This means that $S' = \partial V$ becomes a Heegaard surface for the genus two handlebodies $\overline{S^3 - N(K \cup t_1)}$ and $\overline{S^3 - N(K \cup t_2)}$. By [8], there is at most one Heegaard splitting of a handlebody of a given genus. This implies S' is a common stabilization of S_1 and S_2 and shows a validity of Conjecture 1.1. There are examples of knots having this property — torus knots and 2-bridge knots.

In this paper, we give a sufficient condition for tunnel number one links to have this property and show that 2-bridge links satisfy this condition.

2. Examples

2.1. **Torus knots.** A *torus knot* is a knot on the standard torus embedded in S^3. A torus knot can be characterized by two relatively prime integers p and q. $K_{p,q}$ is a torus knot that winds the standard torus p times in meridional direction and q times in longitudinal direction. A torus knot has 3 types of unknotting tunnels t_p, t_q, t_0 (Figure 1) and they are classified in [2].

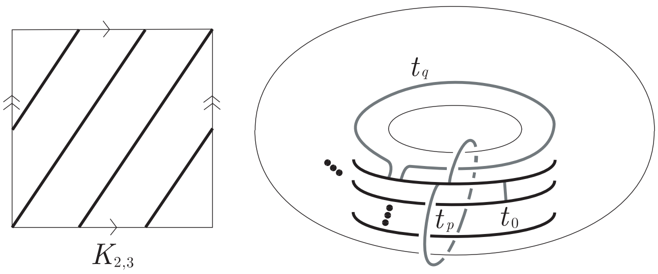

Figure 1. $K_{2,3}$ and unknotting tunnels of a torus knot

Theorem 2.1. [2]*(Boileau-Rost-Zieschang) Let $K_{p,q}$ be a torus knot of type (p,q), where* $\gcd(p,q) = 1$ *and* $p > q > 1$.

(1) *Any unknotting tunnel of $K_{p,q}$ is isotopic to t_p, t_q or t_0.*
(2) *t_0 is isotopic to t_p* $\iff q \equiv \pm 1 \mod p$.
(3) *t_0 is isotopic to t_q* $\iff p \equiv \pm 1 \mod q$.
(4) *t_p is isotopic to t_q* $\iff |p - q| = 1$.

In [2], they also proved that the two Heegaard splittings given by any two unknotting tunnels among t_p, t_q, t_0 of a torus knot, say t_p and t_q, have a common stabilization $S' = \partial N(K \cup t_p \cup t_q)$.

2.2. **2-bridge knots.** S^3 can be understood as a gluing of two 3-balls along the boundary spheres. A 2-*bridge knot* is a knot which can be decomposed into two trivial 2-string tangles in those two 3-balls. A 2-bridge knot has 6 types of unknotting tunnels s_1, s_1', t_1, s_2, s_2', t_2 (Figure 2) and they are classified in [6].

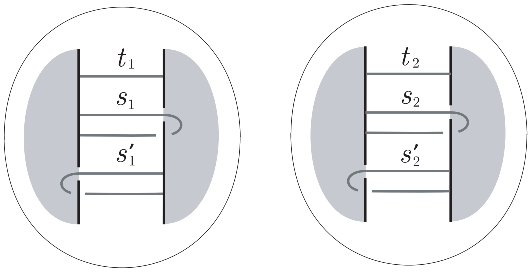

FIGURE 2. Unknotting tunnels of a 2-bridge knot

In [3], Hagiwara showed that the two Heegaard splittings given by any two unknotting tunnels among t_1, s_1, s_1', t_2, s_2, s_2' of a 2-bridge knot have a common stabilization.

3. COMMON STABILIZATIONS OF TUNNEL NUMBER ONE LINK EXTERIOR

When we have two disjoint unknotting tunnels t_1, t_2 of a knot K, $\partial N(K \cup t_1 \cup t_2)$ may not be a Heegaard surface even if t_1 and t_2 are isotopic tunnels. Take t_2 as a parallel copy of t_1. Pull a part of t_2 in a complicated way and hook it to t_1. This construction does not give a genus three Heegaard surface (Figure 3). So there must be some restrictions on the choice of the unknotting tunnels.

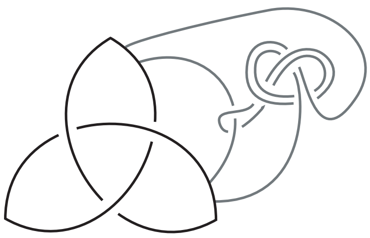

FIGURE 3. Embedding of tunnels which does not give a genus three splitting

A tunnel number one link has two components. If both components are unknotted, it is a 2-bridge link [4]. There are infinitely many tunnel number one links with one knotted component or two knotted components. For example, we can make one with two knotted components from a non-trivial tunnel number one knot by sliding an endpoint of the tunnel over the knot in longitudinal direction many times (Figure 4).

Here we give a sufficient condition for a non-trivial tunnel number one link exterior to have a common stabilization when we have two disjoint unknotting tunnels. Some arguments in the proof work only for 2-component links.

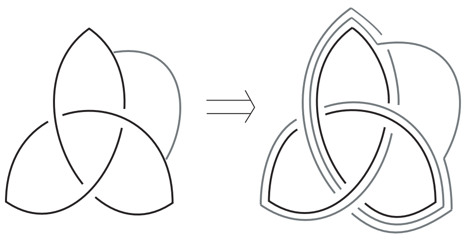

FIGURE 4. Making a tunnel number one link with two knotted components

Theorem 3.1. *Let $L = K_1 \cup K_2$ be a non-trivial tunnel number one link and t_1 and t_2 be two disjoint unknotting tunnels of L such that a meridian disk D of the genus two handlebody $V = \overline{S^3 - N(L \cup t_1)}$ does not intersect t_2. Then $\overline{S^3 - N(L \cup t_1 \cup t_2)}$ is a genus three handlebody.*

4. Proof of Theorem 3.1

Let W be the genus two handlebody $\overline{S^3 - N(L \cup t_2)}$ and E_1 and E_2 be two non-separating meridian disks of W which are not parallel to each other. Then $E_1 \cup E_2$ cuts W into a 3-ball. ∂V (resp. ∂W) consists of three parts as in Figure 5 — two once-punctured tori T_{K_1} and T_{K_2} (resp. T'_{K_1} and T'_{K_2}) and an annulus A_{t_1} (resp. A_{t_2}) joining them.

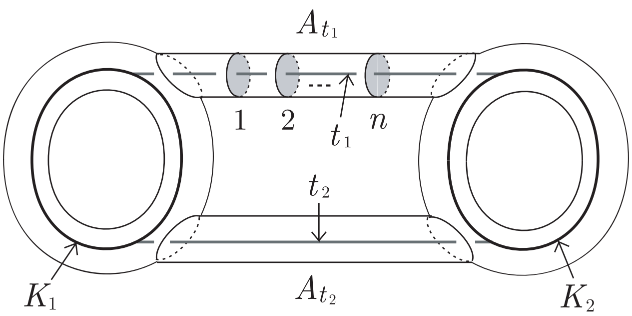

FIGURE 5. $N(L \cup t_1) \cap \mathrm{int}(E_1 \cup E_2)$

Lemma 4.1. *$\partial D \cap A_{t_1}$ and $\partial E_i \cap A_{t_2}$ $(i = 1, 2)$ are non-empty. We may assume that $\partial D \cap T_{K_1}$, $\partial D \cap T_{K_2}$, $\partial D \cap A_{t_1}$ consist of essential arcs and also $\partial E_i \cap T'_{K_1}$, $\partial E_i \cap T'_{K_2}$, $\partial E_i \cap A_{t_2}$ $(i = 1, 2)$ consist of essential arcs.*

Proof. Suppose $\partial D \cap A_{t_1} = \emptyset$. Then ∂D is in $\partial N(K_1)$ or $\partial N(K_2)$, say $\partial D \subset \partial N(K_1)$. If ∂D is a meridian of $\partial N(K_1)$, a punctured $S^2 \times S^1$ is in S^3, a contradiction. If ∂D is a longitude of $\partial N(K_1)$, then the regular neighborhood of $N(K_1) \cup D$ is a 3-ball containing K_1 only. This implies $\overline{S^3 - N(L)}$ is reducible, so L is a non-trivial split link, hence cannot have an unknotting tunnel, a contradiction. If ∂D winds around $\partial N(K_1)$ longitudinally more than once, then a punctured lens space is in S^3, a contradiction. This proves $\partial D \cap A_{t_1} \neq \emptyset$. Similarly $\partial E_i \cap A_{t_2} \neq \emptyset (i = 1, 2)$.

If ∂D meets A_{t_1} in an essential loop, let D' be the meridian disk of t_1 that the essential loop bounds. Then the 2-sphere $D \cup D'$ separates L, so L is a non-trivial split link, hence cannot have an unknotting tunnel, a contradiction. Therefore $\partial D \cap A_{t_1}$ has no essential loops. Similarly $\partial E_i \cap A_{t_2} (i = 1, 2)$ has no essential loops.

If any of $\partial D \cap T_{K_1}$, $\partial D \cap T_{K_2}$, $\partial D \cap A_{t_1}$, $\partial E_i \cap T'_{K_1}$, $\partial E_i \cap T'_{K_2}$, $\partial E_i \cap A_{t_2} (i = 1, 2)$ have inessential arcs, we can remove them by isotopy. So we may assume that all the intersections are essential arcs. □

Label the arcs $\partial D \cap T_{K_1}$, $\partial D \cap T_{K_2}$, $\partial D \cap A_{t_1}$ of ∂D with K_1, K_2, t_1, respectively. Also label the arcs $\partial E_i \cap T'_{K_1}$, $\partial E_i \cap T'_{K_2}$, $\partial E_i \cap A_{t_2}$ of $E_i (i = 1, 2)$ with K_1, K_2, t_2, respectively. Let's assume that t_1 intersects $E_1 \cup E_2$ transversely in n points (possibly n can be zero) and number the meridian disks $N(L \cup t_1) \cap \text{int}(E_1 \cup E_2)$ of t_1 consecutively along $N(L \cup t_1)$ (Figure 5).

Let $|D \cap (E_1 \cup E_2)|$ denote the number of components of $D \cap (E_1 \cup E_2)$. Since D does not intersect t_2 by hypothesis, $D \cap (E_1 \cup E_2)$ consists of loops and properly embedded arcs in D. If there is a circle component of $D \cap (E_1 \cup E_2)$ in D, by cutting and pasting $E_1 \cup E_2$ along an innermost disk in D, we can reduce $|D \cap (E_1 \cup E_2)|$.

So we may assume that $D \cap (E_1 \cup E_2)$ consists of properly embedded arcs in D. An endpoint of arc of $D \cap (E_1 \cup E_2)$ of D in $\partial D \cap A_{t_1}$ corresponds to a meridian disk $N(L \cup t_1) \cap \text{int}(E_1 \cup E_2)$ of t_1. Label that endpoint with the number given to the corresponding meridian disk. Figure 6 shows the intersection $D \cap (E_1 \cup E_2)$ and $D \cap E_i (i = 1, 2)$ on D and $E_i (i = 1, 2)$, respectively.

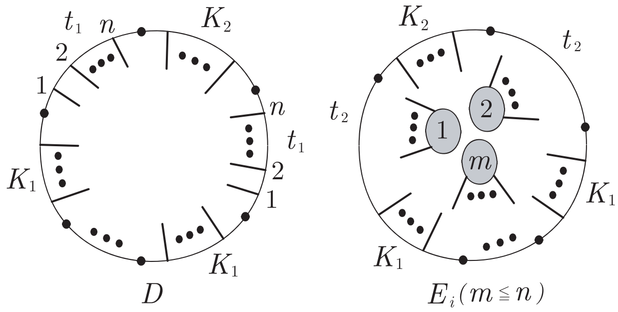

FIGURE 6. $D \cap (E_1 \cup E_2)$ and $D \cap E_i (i = 1, 2)$ with labels

Lemma 4.2. *Any arc of $\partial D \cap T_{K_1}$ and $\partial D \cap T_{K_2}$, which is labeled as K_1 and K_2 respectively, has intersection with $E_1 \cup E_2$.*

Proof. Suppose an arc a of $\partial D \cap T_{K_1}$ has no intersection with $E_1 \cup E_2$. Let b be the subarc of ∂T_{K_1} joining two endpoints of ∂a (Figure 7).

Cut the genus 2-handlebody $W = \overline{S^3 - N(L \cup t_2)}$ by $E_1 \cup E_2$. Then we get a 3-ball. Since $a \cup b$ does not intersect $E_1 \cup E_2$, $a \cup b$ bounds a properly embedded disk in the ball. If $a \cup b$ is a meridian of $\partial N(K_1)$, a punctured $S^2 \times S^1$ is embedded in S^3, a contradiction. If $a \cup b$ is a longitude of $\partial N(K_1)$, then the regular neighborhood of $N(K_1) \cup D$ is a 3-ball containing K_1 only. This implies $\overline{S^3 - N(L)}$ is reducible, so L is a non-trivial split link, hence cannot have an unknotting tunnel, a contradiction. Since a is an essential arc in T_{K_1}, $a \cup b$ winds around $\partial N(K_1)$ longitudinally more than once. Then a punctured lens space is embedded in S^3, a contradiction. Similarly $\partial D \cap T_{K_2}$ should have intersection with $E_1 \cup E_2$. □

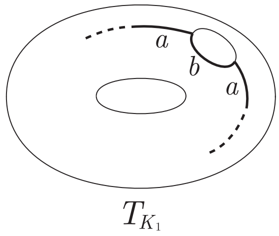

FIGURE 7. $a \cup b$ bounds a disk.

Consider an outermost arc a of $D \cap (E_1 \cup E_2)$ in D. Let Δ be the outermost disk in D corresponding to a. We may assume $a \subset D \cap E_1$. Let $\partial a = \{p, q\}$. By Lemma 4.2 there are 4 cases according to the labels of the arcs of ∂D containing p and q. For the first three cases, we'll show that $|D \cap (E_1 \cup E_2)|$ can be reduced by removing the arc a. In the remaining 4th case we show that there exists a stabilizing disk. Notice that $\partial D \cap (T_{K_1} \cup T_{K_2}) \neq \emptyset$. By Lemma 4.2, we have $|D \cap (E_1 \cup E_2)| > 0$ which guarantees that the case 4 always occurs.

Case 1. p and q are in one arc of $\partial D \cap A_{t_1}$ labeled as t_1, and the two numbers labeled to p and q are i and $i+1$, respectively (Figure 8).

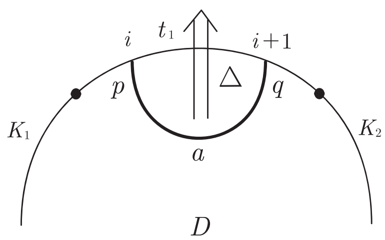

FIGURE 8. p and q are in one arc of $\partial D \cap A_{t_1}$

Pushing E_1 along Δ removes a and so we can reduce $|D \cap (E_1 \cup E_2)|$.

Case 2. p and q are in one arc of $\partial D \cap T_{K_i}$ labeled as $K_i (i = 1, 2)$ (Figure 9).

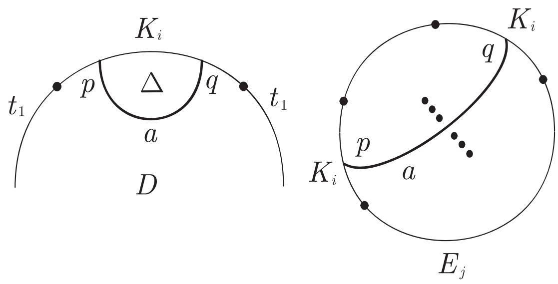

FIGURE 9. p and q are in one arc of $\partial D \cap T_{K_i} (i = 1, 2)$

By cutting and pasting E_1 along Δ, we can get two disks E_1' and E_1''. By pushing slightly, we can make E_1' and E_1'' disjoint from E_1 and E_2.

If E_1' (resp. E_1'') is isotopic to E_1, we can reduce $|D \cap (E_1 \cup E_2)|$ by replacing E_1 with E_1' (resp. E_1''). This occurs when E_1'' (resp. E_1') is ∂-parallel. Suppose that E_1' and E_1'' are not ∂-parallel and not isotopic to E_1. If E_1' (resp. E_1'') is isotopic to E_2, then $E_1'' \cup E_2$ (resp. $E_1' \cup E_2$) cuts W into a 3-ball. Therefore $|D \cap (E_1 \cup E_2)|$ can be reduced by replacing E_1 with E_1'' (resp. E_1'). Now we consider the case that none of E_1' and E_1'' is isotopic to $E_i (i = 1, 2)$. There are two types of essential disks in W which are not isotopic to E_1 or E_2 — separating and non-separating. See Figure 10.

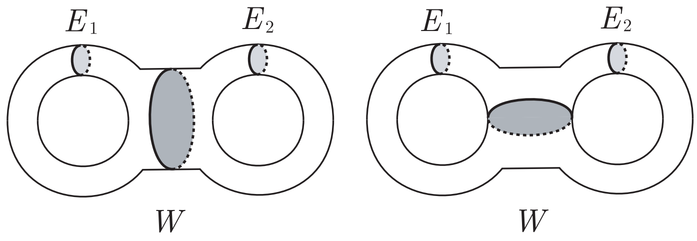

FIGURE 10. Two essential disks in W not isotopic to $E_i (i = 1, 2)$

Since E_1' and E_1'' can be chosen so that $\partial E_1'$ and $\partial E_1''$ are boundary components of $N(E_1 \cup \Delta) \cap \partial W$ which is a pair of pants whose third boundary component is isotopic to ∂E_1, at least one of E_1' and E_1'', say E_1', is non-separating. Then we can reduce $|D \cap (E_1 \cup E_2)|$ by replacing E_1 with E_1'.

Case 3. p is in an arc of $\partial D \cap T_{K_i} (i = 1, 2)$ and q is in an adjacent arc of $\partial D \cap A_{t_1}$ (Figure 11).

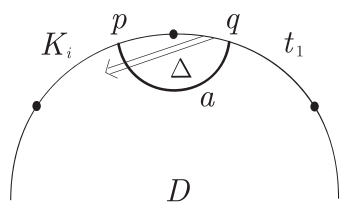

FIGURE 11. p is in an arc of $\partial D \cap T_{K_i} (i = 1, 2)$ and q is in an adjacent arc of $\partial D \cap A_{t_1}$

Note that the number labeled to q is 1 if $i = 1$ and n if $i = 2$. Suppose the number labeled to q is 1. Then the meridian disk of t_1 with label 1 cuts the tubular neighborhood of t_1 in $N(L \cup t_1)$ into two parts. Slide the neighborhood of one of the parts adjacent to Δ along and then off Δ. Then $|D \cap (E_1 \cup E_2)|$ is reduced. It is similar in the case that the number labeled to q is n.

Case 4. p is in an arc of $\partial D \cap T_{K_1}$ and q is in an arc of $\partial D \cap T_{K_2}$, where there is one arc of $\partial D \cap A_{t_1}$ between them (Figure 12).

In this case t_1 does not intersect $E_1 \cup E_2$. By cutting and pasting $E_1 \cup E_2$ along Δ, we get a stabilizing disk of t_1 for the genus two Heegaard surface $\partial N(L \cup t_2)$. So $S^3 - N(L \cup t_1 \cup t_2)$ is a genus three handlebody.

This completes the proof of Theorem 3.1.

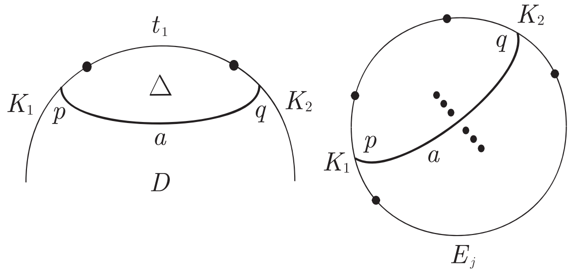

FIGURE 12. p is in an arc of $\partial D \cap T_{K_1}$ and q is in an arc of $\partial D \cap T_{K_2}$ with one arc of $\partial D \cap A_{t_1}$ between them

5. 2-BRIDGE LINKS

Let $S_i = s_i \cup s_i'$ be a trivial 2-string tangle in a 3-ball $B_i (i = 1, 2)$. Gluing B_1 and B_2 along their boundary spheres so that $s_1 \cup s_2$ and $s_1' \cup s_2'$ are simple closed curves, we obtain a 2-bridge link $L = S_1 \cup S_2$. We assume that L is non-trivial. By [4], there are two types of unknotting tunnels for 2-bridge links. If two unknotting tunnels of L are parallel, we can easily find a stabilizing disk. So assume that the two tunnels are in standard positions t_1 and t_2 as in Figure 13.

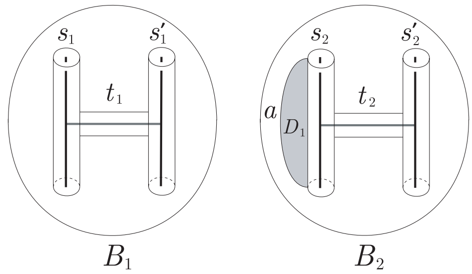

FIGURE 13. Unknotting tunnels of a 2-bridge link

Note that $\overline{B_2 - N(S_2)}$ is a genus two handlebody, and $\overline{B_1 - N(S_1 \cup t_1)}$ is homeomorphic to $(\overline{B_1 - N(S_1 \cup t_1)} \cap \partial B_1) \times I$. Let $f : \partial B_1 \to \partial B_2$ be the gluing homeomorphism.

Let D_1 be the meridian disk as in Figure 13 and $a = \partial D_1 \cap \partial B_2$. Let $b = f(a)$ and D_2 be a disk in $\overline{B_1 - N(S_1 \cup t_1)}$ corresponding to $b \times I \subset (\overline{B_1 - N(S_1 \cup t_1)} \cap \partial B_1) \times I$. Then $D = D_1 \cup_f D_2$ is a meridian disk of genus two handlebody $\overline{S^3 - N(S_1 \cup S_2 \cup t_1)}$ that does not intersect t_2. So this is an example that satisfies the hypothesis of Theorem 3.1 and t_1 and t_2 give a common stabilization.

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