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REIDEMEISTER TORSION OF HOMOLOGY LENS SPACES

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Abstract. We investigate the Reidemeister torsion of a homology lens space obtained by p/q-surgery along a knot K in a homology 3-sphere Σ. We denote it by $\Sigma(K;p/q)$. Firstly we consider the case that the Alexander polynomial of K is the same as that of a torus knot, and give a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K;p/q)$ to be that of a lens space. Secondly we consider the case that the Alexander polynomial of K is of degree 2, and show that if the Reidemeister torsion of $\Sigma(K;p/q)$ is the same as that of a lens space, then $\Delta_K(t) = t^2 - t + 1$. We will talk only the first result and its generalization.

1. Introduction

In 1935, K. Reidemeister [16] has completely classified 3-dimensional lens spaces by using a torsion invariant for closed 3-dimensional PL-manifolds. The invariant is called the *Reidemeister torsion*, which is not a homotopy invariant but a simple-homotopy invariant. He showed that two 3-dimensional lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q \equiv \pm q' \pmod p$ or $qq' \equiv \pm 1 \pmod p$. Later, W. Franz [4] has extended Reidemeister's result to higher dimensional lens spaces by showing an algebraic number theorical result. For alternating proofs, see [1], [15]. In 1962, J. Milnor [11] pointed out that the Reidemeister torsion was closely related to the Alexander polynomial, and V. G. Turaev [20], [21] calculated the Reidemeister torsion of a compact 3-manifold. In 1984, T. Sakai [18] obtained the same result in the case of a homology lens space which is the result of a rational surgery along a knot in S^3. In the talk, we define the Reidemeister torsion following V. G. Turaev [20], [21], [22].

In 1971, L. Moser [13] has determied a homology lens space which is the result of a rational surgery along a torus knot. C. McA. Gordon [7] extended his result to satellite knots. M. Shimozawa [19] reproved Moser's result by using the Kirby-Rolfsen move.

Theorem 1.1. (Moser [13]; Gordon [7]; Shimozawa [19]) *Let $K_{r,s}$ be the (r,s)-torus knot in S^3, and $M = S^3(K_{r,s};p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|, |r|, |s| \geq 2$ and $q \neq 0$. Then there are three cases :*

(1) *If $|p - qrs| \neq 0$, then M is a Seifert fibered space with three singular fibers of multiplicities $|r|, |s|$ and $|p - qrs|$. In particular,*

(2) *if $|p - qrs| = 1$, then M is the lens space $L(p, qr^2)$* (Figure 1).

(3) *If $|p - qrs| = 0$ $(p/q = rs)$, then M is the connected sum of two lens spaces, $L(r,s)\sharp L(s,r)$.*

For example, $M = S^3(K_{2,3}; 5)$ is homeomorphic to $L(5, -1)$ by the theorem above (Figure 1).

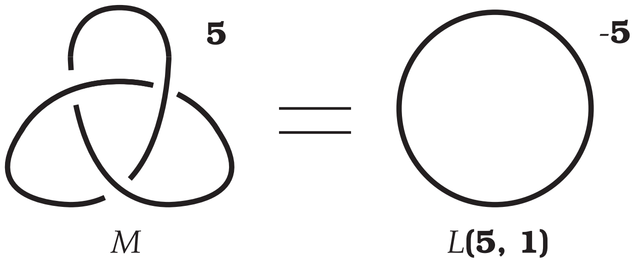

FIGURE 1. the result of surgery is a lens space

We define that an algebraic number is *of lens space type* if it is the Reidemeister torsion of a certain lens space, and that an orientable closed 3-manifold is *of lens space type* if its Reidemeister torsion over any coefficient is of lens space type. We denote

$$\Delta_{r,s}(t) := \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)} \qquad ((r, s) = 1).$$

The following is the first main theorem.

Main Theorem 1 *Let $K_{r,s}$ $((r, s) = 1)$ be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_{r,s}(t)$, and $M = \Sigma(K_{r,s}; p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|, |r|, |s| \geq 2$ and $q \neq 0$. Then M is of lens space type if and only if the following (1) and (2) hold.*

(1) $(p, r) = 1$, *and* $(p, s) = 1$, *and*

(2) $r \equiv \pm 1 \pmod{p}$ *or* $s \equiv \pm 1 \pmod{p}$ *or* $qrs \equiv \pm 1 \pmod{p}$.

We prove this theorem by using Franz's Theorem [3], [4]. We notice that the Reidemeister torsion can distinguish between some homology lens space and a lens space. However it is not complete. For example, though $S^3(K_{2,3}; -5), S^3(K_{2,3}; -7)$ and $S^3(K_{3,5}; 4)$ are of lens space types from our result, they are not lens spaces by Moser's Theorem.

H. Goda and M. Teragaito [6] considered when a rational surgery along a hyperbolic knot yields a lens space. The following is a part of their results.

Theorem 1.2. (Goda-Teragaito [6]) *Let K be a genus 1 knot in S^3. If a rational surgery along K yields a lens space, then K is the trefoil.*

P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó [10] extended the result above to genus up to 5.

We denote

$$\Delta_n(t) := n(t - 1)^2 + t = nt^2 - (2n - 1)t + n \quad (n \neq 0).$$

Corresponding to Goda-Teragaito's result, we obtained the following.

Main Theorem 2 *Let K be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_K(t) = \Delta_n(t)$, and $M = \Sigma(K; p/q)$ the result of p/q-surgery along K*

where $|p| \geq 2$ *and* $q \neq 0$*. Let* ξ_d *be a primitive* d*-th root of unity,* $\psi_d : \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\xi_d)$ *a homomorphism such that* $\psi_d(t) = \xi_d$ *where* d (≥ 2) *is a divisor of* p*, and* $\tau^{\psi_d}(M)$ *the Reidemeister torsion associated to* ψ_d*. Then the following* (1) *and* (2) *holds.*

(1) *If* $n \leq -1$*, then* $\tau^{\psi_p}(M)$ *is not of lens space type.*

(2) *If* $|n| \geq 2$ *and* d *is a prime number, then* $\tau^{\psi_d}(M)$ *is of lens space type.*

The result leads to the following corollary.

Corollary 1.3. *In the same situation as Main Theorem 2, if* M *is of lens space type, then*

$$\Delta_K(t) = t^2 - t + 1 \quad (n = 1).$$

P. Ozsváth and Z. Szabó [14] obtained the following theorem by using the knot Floer homology.

Theorem 1.4. (Ozsváth-Szabó [14]) *Let* K *be a knot in* S^3*, and* $M = S^3(K; p)$ *the result of* p*-surgery along* K *where* p *is an integer. If* M *is a lens space, then the Alexander polynomial of* K *is the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{s_j} + t^{-s_j}),$$

where $0 < s_1 < s_2 < \cdots < s_m$*.*

Our result extends a special case of Ozsváth-Szabó's result.

We raise the following question.

Question 1.5. *If* $\Sigma(K; p/q)$ *is a lens space, then is* $\Delta_K(t)$ *a product of cyclotomic polynomials ?*

R. Fintushel and R. J. Stern [5] showed thet 18 and 19-surgery along the $(-2, 3, 7)$-pretzel knot are lens spaces. The Alexander polynomial of the $(-2, 3, 7)$-pretzel knot K is

$$\Delta_K(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1.$$

This is an irreducible polynomial, and is not a product of cyclotomic polynomials. This is a counterexample for Question 1.5.

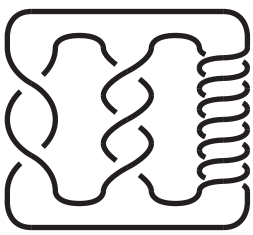

FIGURE 2. $(-2, 3, 7)$-pretzel knot

In the talk, we prove only Main Theorem 1, and explain its generalization. For the proof of Main Theorem 2 and its generalization, see [9].

2. Reidemeister torsion of homology lens spaces

For the definition of Reidemeister torsion and its properties, see [20], [21], [22]. The following theorem is an important tool for our theorems.

Theorem 2.1. (Turaev [20], [21], [22]; Sakai [18]) *Let K be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_K(t)$, and $M = \Sigma(K; p/q)$ the result of p/q-surgery along K where $|p| \geq 2$ and $q \neq 0$. Let $\psi_d : \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\xi)$ be a homomorphism such that $\psi_d(t) = \xi$ where ξ is a primitive d-th root of unity, and d (≥ 2) is a divisor of p. Then the Reidemeister torsion of M, $\tau^{\psi_d}(M)$, associated to ψ_d is*

$$\tau^{\psi_d}(M) = \Delta_K(\xi)(\xi - 1)^{-1}(\xi^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod{p}$.

The lens space $L(p, q)$ is the case that a knot K is the trivial knot in S^3.

Theorem 2.2. (Reidemeister [16]; Franz [4])

$$\tau^{\psi_d}(L(p, q)) = (\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod{p}$.

Lens spaces are classified by using the Reidemeister torsion.

Theorem 2.3. (Reidemeister [16]; Franz [4]; Brody [1]; Przytycki-Yasuhara [15]) *Two lens spaces $L(p, q)$ and $L(p', q')$ are homeomorphic if and only if (1) $p = p'$, and (2) $q \equiv \pm q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$.*

To prove Theorem 2.3, an algebraic number theorical result, due to Franz [4], is applied.

Theorem 2.4. (Franz [4]) *Let ζ be a primitive n-th root of unity, S the set of non-zero divisors in $\mathbf{Z}/n\mathbf{Z}$. Let $\{a_j \ (j \in S)\}$ be integers satisfying the following conditions:*

$$(1)\ a_{-j} = a_j, \quad (2) \sum_{j \in S} a_j = 0, \quad (3) \prod_{i \in S} (\zeta^i - 1)^{a_i} = 1.$$

Then $a_j = 0$ for all $j \in S$.

We use Theorem 2.4 to prove Main Theorem1.

3. Proof of Main Theorem 1

Let d be a divisor of p with $d \geq 2$, ξ a primitive d-th root of unity, and ψ_d : $\mathbf{Z}[t]/(t^p - 1) \to \mathbf{Q}(\xi)$ a homomorphism such that $\psi_d(t) = \xi$.

Case 1 $(p, r) = 1$ and $(p, s) = 1$.

By Theorem 2.1,

$$\tau^{\psi_d}(M) = \frac{(\xi^{rs} - 1)(\xi - 1)}{(\xi^r - 1)(\xi^s - 1)} \cdot (\xi - 1)^{-1}(\xi^{\bar{q}} - 1)^{-1}.$$

If $\tau^{\psi_d}(M)$ is lens space type, then there are integers i, j and m such that

$$\tau^{\psi_d}(M) = \pm \xi^m (\xi^i - 1)^{-1}(\xi^j - 1)^{-1},$$

where $(i, d) = 1,\ (j, d) = 1$.

If $\tau^{\psi_d}(M)$ is lens space type, then

$$(\xi^{rs}-1)(\xi^i-1)(\xi^j-1)=\pm\xi^m(\xi^{\bar{q}}-1)(\xi^r-1)(\xi^s-1).$$

Multipling the complex conjugates to both sides,

$$(\xi^{rs}-1)(\xi^i-1)(\xi^j-1)(\xi^{-rs}-1)(\xi^{-i}-1)(\xi^{-j}-1)$$
$$=(\xi^{\bar{q}}-1)(\xi^r-1)(\xi^s-1)(\xi^{-\bar{q}}-1)(\xi^{-r}-1)(\xi^{-s}-1).$$

By Franz's Theorem,

$$\{\pm rs,\ \pm i,\ \pm j \pmod d\}=\{\pm\bar{q},\ \pm r,\ \pm s \pmod d\}.$$

(i) $rs\equiv\pm s \pmod d$ or $rs\equiv\pm r \pmod d$.

This is equivalent to $r\equiv\pm1 \pmod d$ or $s\equiv\pm1 \pmod d$.

(ii) $rs\equiv\pm\bar{q} \pmod d$.

This is equivalent to $qrs\equiv\pm1 \pmod d$.

If (i) or (ii) holds for p, then the same condition holds for any d. So the condition that (i) or (ii) holds for all d is equivalent to the condition that (i) or (ii) holds for p.

Case 2 $(p,r)=d\geq 2$. $\quad((d,s)=1)$

Let $p=p'd$ and $r=r'd$.

$$\Delta_{r,s}(t)=\frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)}=\frac{\dfrac{t^{rs}-1}{t^d-1}\cdot(t-1)}{\dfrac{t^r-1}{t^d-1}\cdot(t^s-1)}$$

$$=\frac{t^{(r's-1)d}+t^{(r's-2)d}+\cdots+t^d+1}{t^{(r'-1)d}+t^{(r'-2)d}+\cdots+t^d+1}\cdot\frac{t-1}{t^s-1}.$$

$$\tau^{\psi_d}(M)=\frac{s}{(\xi^{\bar{q}}-1)(\xi^s-1)}.$$

Let $K/\mathbf{Q}$ be a finite Galois extension, and α an element of K. We denote the norm of α over $\mathbf{Q}$ by $N_{K/\mathbf{Q}}(\alpha)$. Since

$$N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\pm\xi^m)=1,$$

$$N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\xi^{\bar{q}}-1)=N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\xi^s-1)=N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\xi^i-1)=N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\xi^j-1)\neq 0$$

and

$$N_{\mathbf{Q}(\xi)/\mathbf{Q}}(s)=s^{\varphi(d)}\quad(|s|\geq 2),$$

where $\varphi(d)$ is the Euler function,

$$N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\tau^{\psi_d}(M))\neq N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\pm\xi^m(\xi^i-1)^{-1}(\xi^j-1)^{-1}).$$

This means $\tau^{\psi_d}(M)$ is not of lens space type.

Therefore only Case 1 is adopted. □

4. Genaralization of Main Theorem 1

The following is a generalization of Main Theorem 1.

Theorem 4.1. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial is $\Delta_K(t)$, and ζ a primitive p-th root of unity where $p \geq 2$. If the following two condition (1) and (2) holds:*

(1) $$\Delta_K(\zeta) = \frac{\prod_{k=1}^{l}(\zeta^{v_k} - 1)}{\prod_{k=1}^{l}(\zeta^{u_k} - 1)}$$

where $(u_k, p) = 1$ and $(v_k, p) = 1$ $(k = 1, \cdots, l)$, and $\{\pm u_k \ (\mathrm{mod} p)\}$ and $\{\pm v_k \ (\mathrm{mod} p)\}$ does not have common numbers,

(2) $\tau^{\psi_p}(\Sigma(K; p/q))$ $(q \neq 0)$ *is of lens space type,*

then we have $l \leq 2$.

Corollary 4.2. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial is $\Delta_K(t) = (t^2 - t + 1)^m$ $(m \geq 2)$. Then any rational surgery along K cannot be of lens space type.*

Let $\mathbf{\Phi}_n(x)$ be the n-th cyclotomic polynomial. That is, let ζ be a primitive n-th root of unity. Then

$$\mathbf{\Phi}_n(x) = \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (x - \zeta^i).$$

Corollary 4.3. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial is $\Delta_K(t) = \mathbf{\Phi}_{30}(t)$. If the p/q-surgery along K $(p \geq 2, q \neq 0)$ is of lens space type, then $p = 2, 3$ or 4.*

We notice that $\mathbf{\Phi}_{30}(t)$ does not satisfy the condition in Theorem 1.4.

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