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BRAIDS, HYPERGEOMETRIC INTEGRALS AND CONFORMAL FIELD THEORY

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1. Introduction

Let X_n denote the configuration space of ordered distinct n points on the complex plane and $\pi : X_{n+m} \to X_n$ the projection map on the first n factors. Let us consider a rank one local system $\mathcal{L}$ on X_{m+n}. The action of the braid group B_n on the m-th homology of a fiber of π with coefficients in $\mathcal{L}$ provides a series of linear representations of B_n. Such representations of the braid groups might be understood as the action of braids on hypergeometric integrals. In the simplest case $m = 1$ we obtain the Burau representation. On the other hand, the mopnodromy representations of the braid groups appearing in conformal field theory have been studied extensively in relation with quantum groups. The purpose of this note is to clarify the relation between the representations of the braid groups appearing as the action on hypergeometric integrals and the monodromy representations of conformal field theory. We show that a basis of the space of conformal blocks is given by hypergeometric integrals where the cycles of integrations are regularized bounded chambers. We also describe how the Jones representations and the Lawrence-Krammer representations arise in this framework. For details we refer the readers to [3].

2. Conformal blocks

We first recall the definition and basic properties of the space of conformal blocks. We refer the readers to [4] and [2] for details. In this article we deal with the case the Lie algebra $\mathfrak{g}$ is $sl(2, \mathbf{C})$. Let λ be a nonnegative integer and denote by V_λ the irreducible representation of $\mathfrak{g}$ with highest weight λ. We have $\dim V_\lambda = \lambda + 1$. The affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c$$

is the canonical central extension of the loop algebra $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$. In the following we fix a positive integer k. For an integer λ with $0 \leq \lambda \leq k$ we denote by $\mathcal{H}_\lambda$ the integrable highest weight module of $\widehat{\mathfrak{g}}$ with highest weight λ and central charge k. It is an irreducuble quotient of the Verma module M_λ by the unique maximal submodule generated by a null vector. Let $p_1, \cdots, p_n, p_{n+1}$ be distinct points on $\mathbf{C}P^1$ with $p_{n+1} = \infty$. We denote by $\mathcal{M}$ the space of algebraic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_n, p_{n+1}$. To each point $p_1, \cdots, p_n, p_{n+1}$ we associate integers $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ with $0 \leq \lambda_j \leq k$, $1 \leq j \leq n+1$. Then the Lie algebra $\mathfrak{g} \otimes \mathcal{M}$ acts diagonally on the tensor product $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_{n+1}}$ by the Laurent expansion at $p_1, \cdots, p_n, p_{n+1}$. The space of conformal blocks $\mathcal{H}(p, \lambda)$ is defined to by the space of coinvariant tensors

$$\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_{n+1}} / \mathfrak{g} \otimes \mathcal{M}$$

by means of the above diagonal action. Its dual space is defined as the space of invariant multilinear maps

$$\mathcal{H}(p,\lambda)^* = \mathrm{Hom}_{\mathfrak{g}\otimes\mathcal{M}}(\mathcal{H}_{\lambda_1}\otimes\cdots\otimes\mathcal{H}_{\lambda_{n+1}}, \mathbf{C})$$

It can be shown that the restriction map

$$i : \mathcal{H}(p,\lambda)^* \to \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_{n+1}}, \mathbf{C})$$

is injective. The space of conformal blocks $\mathcal{H}(p,\lambda)$ is a quotient of the space of coinvariant tensors $(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_{n+1}})_{\mathfrak{g}}$ and is a finite dimensional vector space.

The space of conformal blocks $\mathcal{H}(p,\lambda)$ forms a vector bundle over the configuration space

$$X_n = \{(z_1,\cdots,z_n)\in\mathbf{C}^n \mid z_i\neq z_j \text{ if } i\neq j\}$$

equipped with a flat connection called the KZ connection.

3. Twisted de Rham complex

In the following, we deal with the case

$$m = \frac{1}{2}(\lambda_1+\cdots+\lambda_n-\lambda_{n+1}).$$

Let $\pi : X_{n+m}\to X_n$ denote the projection map defined by

$$\pi(z_1,\cdots,z_n;t_1,\cdots,t_m) = (z_1,\cdots,z_n).$$

Denote by $X_{n,m}$ a fiber of the projection map $\pi : X_{n+m}\to X_n$. We define a multivalued function $\varphi(z,t)$ over X_{n+m} as

$$\varphi(z,t) = \prod_{1\le i\le m, 1\le \ell\le n}(t_i-z_\ell)^{-\frac{\lambda_i}{\kappa}}\prod_{1\le i<j\le m}(t_i-t_j)^{\frac{2}{\kappa}}$$

where we set $\kappa = k+2$.

We denote by $\mathcal{L}$ the rank one local system on $X_{n,m}$ associated with the multivalued function $\varphi(z,t)$. Since $\varphi(z,t)$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ by the permutation of coordinates $(t_1,\cdots,t_m)$, it defines a multivalued function on the quotient space $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$. Let us now consider the twisted de Rham complex of smooth differential forms $(\Omega^*(Y_{n,m}),\nabla)$ over $Y_{n,m}$ with the covariant derivative $\nabla : \Omega^k(Y_{n,m})\to\Omega^{k+1}(Y_{n,m})$ defined by

$$\nabla\omega = d\omega + d\log\varphi\wedge\omega$$

for $\omega\in\Omega^k(Y_{n,m})$. The cohomology of the above twisted de Rham complex is denoted by $H^*(\Omega^*(Y_{n,m}),\nabla)$. Let $\mathcal{L}^*$ denote the dual local system of $\mathcal{L}$. There is a hypergeometric pairing

$$H_m(Y_{n,m},\mathcal{L}^*)\times H^m(\Omega^*(Y_{n,m}),\nabla)\to\mathbf{C}$$

defined by

$$(c,\omega)\mapsto\int_c\varphi\omega.$$

If the parameters $\lambda_1,\cdots,\lambda_n$ and k are generic, we have the vanishing theorem claming that

$$H_j(Y_{n,m},\mathcal{L}^*)\cong 0$$

unless $j = m$ and that the middle dimensional homology $H_m(Y_{n,m}, \mathcal{L}^*)$ is isomorphic to $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$, the homology with locally finite chains. In this case $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$ is spanned by bounded chambers. In our case, $\lambda_1, \cdots, \lambda_n$ are nonnegative integers and k is a positive integer, and the above vanishing theorem does not necessarily hold. The purpose of our study is to clarify the cycles of integrations in the case of such special parameters.

It was shown by Feigin, Varchenko and Schechtman [1] that there is a map

$$\rho : (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}})_{\mathfrak{g}} \to \Omega^m(Y_{n,m})$$

of the form

$$\rho(v) = R_v(z,t) dt_1 \wedge \cdots \wedge dt_m$$

with rational functions $R_v(z,t)$ inducing an injective homomorphism

$$\rho_* : \mathcal{H}(p,\lambda) \to H^m(\Omega^*(Y_{n,m}), \nabla).$$

Here we have an isomorphism

$$H^m(\Omega^*(Y_{n,m}), \nabla) \cong H^m(Y_{n,m}, \mathcal{L}).$$

The dual construction gives a period map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}(p,\lambda)^*$$

by associating to a cycle c the map

$$v \mapsto \int_c \varphi \rho(v)$$

for a coinvariant tensor $v \in (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}})_{\mathfrak{g}}$.

4. Regularizable cycles

There is a narutal map

$$\alpha : H_m(Y_{n,m}, \mathcal{L}^*) \to H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

and the image of α is called the set of regularizable cycles. We denote Im α by $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$. There is an isomorphism

$$H_m(Y_{n,m}, \mathcal{L}^*)/\text{Ker } \alpha \cong H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}.$$

It can been shown that the set of regularizable cycles $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$ is generated by bounded chambers, but it is important to notice that there might be nontrivial relations among them. Our main result is as follows.

Theorem 4.1. *The period map ϕ induces an isomorphism*

$$H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p,\lambda)^*.$$

Let $i : Y_{n,m} \to \text{Sym}^m \mathbf{C}P^1$ be the inclusion map, where $\text{Sym}^m \mathbf{C}P^1$ denotes the m-fold symmetric product. We have the Leray spectral sequence with

$$E_2^{p,q} = H^p(\text{Sym}^m \mathbf{C}P^1, R^q i_* \mathcal{L})$$

converging to $E_\infty^{p,q} = H^{p+q}(Y_{n,m}, \mathcal{L})$. In [4] Silvotti conjectures that the image of $\rho_* : \mathcal{H}(p,\lambda) \to H^m(Y_{n,m}.\mathcal{L})$ is equal to $E_\infty^{m,0}$. Considering the dual of our main theorem, we confirm that Silvotti's conjecture is true.

The case $n = 2$ describes the basic fusion rule for the dimension of the space of conformal blocks for $\mathbf{C}P^1$ with 3 punctures. In fact we have

$$H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg} \cong \mathbf{C}$$

if and only if the quantum Clebsch-Gordan condition is satisfied. Otherwise,

$$H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg} \cong 0.$$

Here the quantum Clebsch-Gordan condition is

$$\begin{gathered} |\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbf{Z} \\ \lambda_1 + \lambda_2 + \lambda_3 \leq 2k. \end{gathered}$$

Let us consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$. The braid group B_n acts on $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$. This representation of the braid group in the case $m = 2$ is called the Lawrence-Krammer representation and was shown to be faithful for generic parameters λ and k by Bigelow. In our case we have a subrepresentation of B_n on $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$ describing the monodromy of the space of conformal blocks. In particular, in the case $\lambda = 1$, this construction gives irreducible and uniraty representations of the braid groups due to Jones.

References

[1] B. Feigin, V. Schechtman and A. Varchenko, *On algebraic equations satisfied by hypergeometric correlators in WZW models. I*, Commun. Math. Phys, **163** (1994), 173–184.

[2] T. Kohno, *Conformal Field Theory and Topology*, Iwanami Series in Modern Mathematics, Translations of Mathematical Monographs, A.M.S., **210** (2002).

[3] T. Kohno, *Hypergeometric integrals and the space of conformal blocks*, preprint (2004).

[4] R. Silvotti, *Local systems on the complement of hyperplanes and fusions rules in conformal field theory*, IMRN **1** (1994), 111–128.

[4] A. Tsuchiya and Y. Kanie, *Vertex operators in conformal field theory on* $\mathbf{P}^1$ *and monodromy representations of braid groups*, Advanced Studies in Pure Math., **16** (1988), 297–372.

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