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4-DIMENSIONAL SURGERY ON A "POCHETTE"

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Let F_2 denote the free group with two generators. The following theorem is well known and dates back to Nielsen [8]:

Theorem 1.

$$\mathrm{Aut}(F_2)/\mathrm{Inn}(F_2) \cong GL_2(\mathbb{Z}) \tag{1}$$

In this paper, we will discuss a new type of 4-dimensional surgery performed along an embedded "pochette", in which the above theorem plays a key role.

1. Pochette surgery

Definition. A 4-manifold diffeomorphic to $S^1 \times D^3 \natural D^2 \times S^2$ is called a *pochette* (a bag with a shoulder strap). A pochette P has the same homotopy type as $S^1 \vee S^2$, and this motivated us to choose the name "pochette".

Embed a pochette P into a 4-manifold M smoothly by $f : P \to M$. Remove the interior $\mathrm{Int} f(P)$ and paste it back via a diffeomorphism $h : \partial P \to \partial(M \setminus \mathrm{Int} f(P))$. We call this process a *pochette surgery* on M along $f(P)$. Simple loops l and m on ∂P corresponding to $S^1 \times \{*\}$ and $\partial D^2 \times \{*\}$ are called a *longitude* and *meridian*, respectively.

Let $M(f, h)$ be the 4-manifold obtained by a pochette surgery on M using an embedding $f : P \to M$ and a diffeomorphism $h : \partial P \to \partial(M \setminus \mathrm{Int} f(P))$. By changing the orientation of P if necessary, we may asuume that the diffeomorphism h is orientation preserving.

Our first result is the following:

Theorem 2. *The diffeomorphism type of the resulting 4-manifold $M(f, h)$ is determined by the embedding f, the homology class $h_*([m]) \in H_1(\partial(M \setminus \mathrm{Int} f(P)); \mathbb{Z}) \cong \mathbb{Z}[m] \oplus \mathbb{Z}[l]$ and a modulo 2 framing around $h(m)$.*

The boundary ∂P is diffeomorphic to $S^1 \times S^2 \# S^1 \times S^2$. To prove Theorem 2, let us recall a theorem on the isotopy classes of the diffeomorphisms $S^1 \times S^2 \# S^1 \times S^2$.

Theorem 3. (Laudenbach [6]) *The sequence*

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \pi_0(\mathrm{Diff}^+(S^1 \times S^2 \# S^1 \times S^2,\ \mathrm{mod}\ x_0)) \to \mathrm{Aut}(F_2) \to 1 \tag{2}$$

is exact. Here $\pi_1(S^1 \times S^2 \# S^1 \times S^2, x_0)$ is identified with F_2 generated by l and m.

Let $\rho : S^1 \times S^2 \to S^1 \times S^2$ be the diffeomorphism sending $(e^{i\theta}, (e^{i\varphi}, h)) \in \mathbb{C} \times (\mathbb{C} \times \mathbb{R})$ to $(e^{i\theta}, (e^{i(\theta+\varphi)}, h)) \in \mathbb{C} \times (\mathbb{C} \times \mathbb{R})$. Let ρ_1 (resp. ρ_2) : $S^1 \times S^2 \# S^1 \times S^2 \to S^1 \times S^2 \# S^1 \times S^2$ be a diffeomorphism which "gives" ρ on the first (resp.the second) summand of $S^1 \times S^2 \# S^1 \times S^2$ and the identity on the other summand. Then the isotopy classes of ρ_1 and ρ_2 generate the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the sequence (2).

The diffeomorphism ρ_1 extends to a self-diffeomorphism of $S^1 \times D^3 \natural D^2 \times S^2$, but ρ_2 does not. The map $\rho_1 \circ \rho_2$ does not extend either. Since $\pi_1(SO_3) \cong \mathbb{Z}_2$, the map $(\rho_2)^2$ extends.

For notational simplicity, we denote the pochette $S^1 \times D^3 \natural D^2 \times S^2$ by P, and the meridian disk $D^2 \times \{*\}$ embedded in the second summand of P by D_m. We assume additionally (without loss of generality) that ρ_2 pointwise fixes the meridian curve $m(= \partial D_m)$.

Lemma 4. *The diffeomorphism type of $M(f,h)$ is determined by M, $f : P \to M$ and the isotopy class of the simple closed curve $h(m)$ and its* mod *2 framing. .*

Proof. Let $h_1, h_2 : \partial P \to \partial(M \setminus \mathrm{Int} f(P))$ be diffeomorphisms such that $h_1(m)$ and $h_2(m)$ are isotopic, and such that the mod 2 framings along these curves are the same. We may assume $h_1(m) = h_2(m)$ and that the difference of their framings is $2k$. Then the composition $h_2^{-1}h_1$ is isotopic to $(\rho_2)^{2k}$ in a neighborhood of $m = \partial D_n$ in ∂P. The composition $(\rho_2)^{-2k}h_2^{-1}h_1$ extends to a regular neighborhood $N(D_m)$ in P. Since $(\rho_2)^{2k}$ extends to P, $h_2^{-1}h_1$ extends to $N(D_m)$. Since $P \setminus \mathrm{Int}N(D_m)$ is diffeomorphic to $S^1 \times D^3$, the composition $h_2^{-1}h_1$ further extends to a self-diffeomorphism $H : P \to P$ by [7]. Then we can construct a diffeomorphism $\Phi : (M \setminus \mathrm{Int} f(P)) \cup_{h_1} P \to (M \setminus \mathrm{Int} f(P)) \cup_{h_2} P$ by defining $\Phi|(M \setminus \mathrm{Int} f(P)) = id$, and $\Phi|P = H$. □

Lemma 5. *Let h, $h' : \partial P \to \partial P$ be orientation preserving self-diffeomorphisms. Suppose that $h_*[m] = h'_*[m] \in H_1(\partial P;\mathbb{Z})$. Then the simple closed curves $h(m)$ and $h'(m)$ are isotopic to each other.*

Proof. Take a base point x_0 on the meridian curve m. Then m represents a homotopy class $\langle m\rangle \in \pi_1(\partial P, x_0)$. Connecting the longitude l to x_0 by a curve on ∂P, we assume that l also represents a homotopy class $\langle l\rangle \in \pi_1(\partial P, x_0)$. We may assume $h(x_0) = h'(x_0) = x_0$, then h and h' induces automorphisms

$$h_\#, h'_\# : F_2 \to F_2,$$

where $F_2 = \pi_1(\partial P, x_0)$ is generated by $\langle l\rangle$ and $\langle m\rangle$.

Put $g = h'^{-1}h$. Then g induces an automorphism

$$g_* : H_1(\partial P;\mathbb{Z}) \to H_1(\partial P;\mathbb{Z})$$

By the assumption of Lemma, we have

$$\begin{cases} g_*[m] = [m] \\ g_*[l] = p[m] + q[l], \quad p, q \in \mathbb{Z} \end{cases}$$

Since g_* is an automorphism of $H_1(\partial P;\mathbb{Z}) \cong \mathbb{Z}[m] \oplus \mathbb{Z}[l]$, q must be ± 1. It is easy to find an orientaion preserving self-diffeomorphism $\sigma : P \to P$ such that $\sigma_\#(\langle l\rangle) = \langle l\rangle^{\pm 1}$ and such that σ pointwise fixes the meridian curve m.

Then the composition σg induces an automorphism of $H_1(\partial P;\mathbb{Z})$ satisfying

$$\begin{cases} \sigma_* g_*([m]) = [m] \\ \sigma_* g_*([l]) = p[m] + [l], \quad p \in \mathbb{Z} \end{cases}$$

We can find an orientation preserving self-diffeomorphism $\tau : P \to P$ such that $\tau_{\#}(\langle l \rangle) = \langle l \rangle \langle m \rangle^{-p}$ and such that τ pointwise fixes the meridian curve m. In fact, τ slides the 1-handle corresponding to $\langle l \rangle$ round the meridian curve $-p$ times.

Now the composition $\tau\sigma g$ induces the identity on $H_1(\partial P; \mathbb{Z})$. Theorem 1 implies that there exists an element $w \in F_2$ such that

$$(\tau\sigma g)_{\#}(x) = w^{-1}xw, \quad \forall x \in F_2$$

The inner automorphism $I_w(x) = wxw^{-1}$ is realised by an orientation preserving self-diffeomorphism $i_w : \partial P \to \partial P$. The diffeomorphism i_w is isotopic to the identity via an isotopy moving the base point x_0 along a loop homotopic to w. Then the composition

$$i_w \tau\sigma g$$

induces the identity on the free group F_2, and by Theorem 3, we have (in $\pi_0(\mathrm{Diff}^+(S^1 \times S^2 \# S^1 \times S^3))$)

$$i_w \tau\sigma g = (\rho_1)^{\varepsilon_1}(\rho_2)^{\varepsilon_2}, \quad \text{where} \quad \varepsilon_1, \varepsilon_2 \in \{0, \pm 1\} \tag{3}$$

Recall that $g = h'^{-1}h$. By (3), we see that $i_w\tau\sigma h'^{-1}h(m)$ is isotopic to the curve $(\rho_1)^{\varepsilon_1}(\rho_2)^{\varepsilon_2}(m)$. The latter, however, is the curve m itself, because ρ_1 and ρ_2 pointwide fix the curve m.

Therefore, the curve $h(m)$ is isotopic to $h'\sigma^{-1}\tau^{-1}{i_w}^{-1}(m)$.

Now i_w is isotopic to the identity, τ and σ do not move the curve m. Thus $h'\sigma^{-1}\tau^{-1}{i_w}^{-1}(m)$ is freely isotopic to the curve $h'(m)$. This completes the proof of that the curve $h(m)$ is isotopic to $h'(m)$. □

Proof of Theorem 2. Theorem 2 is now a corollary to Lemmas 4 and 5. □

2. Slope

A homology class $p[m] + q[l] \in \mathbb{Z}[m] \oplus \mathbb{Z}[l]$ is determined up to ± 1 by its *slope*

$$\frac{p}{q} \quad \in \quad \mathbb{Q} \cup \{\infty\}.$$

Thus, by Theorem 2, a 4-dimensional pochette surgery on M is determined by an embedding of a pochette $f : P \to M$, a slope $\frac{p}{q}$, and a mod 2 framing. By theorems 1,and 3, we have an onto homomorphism

$$\pi_0(\mathrm{Diff}^+(S^1 \times S^2 \# S^1 \times S^2)) \to GL_2(\mathbb{Z}) \tag{4}$$

Thus given any slope $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$, we can find an orientation preserving diffeomorphism $h : S^1 \times S^2 \# S^1 \times S^2 \to S^1 \times S^2 \# S^1 \times S^2$ such that the homology class $h_*([m])$ corresponds to the slope $\frac{p}{q}$. By Lemma 5, the free isotopy class of such a smooth closed curve $h(m)$ in $S^1 \times S^2 \# S^1 \times S^2$ is determined by the slope $\frac{p}{q}$ up to the direction of the curve. Thus we have a unique "natural lift "of the slope $\frac{p}{q}$ to the free group F_2 (up to an inner automorphism and the anti-automorphism $x \mapsto x^{-1}$) represented by the homotopy class of the smooth curve $h(m)$.

Theorem 6. *If $p > 0, q > 0$, a natural lift of the slope $\frac{p}{q}$ to $F_2 = \langle l, m\rangle$ is*

$$l^{[\frac{q}{p}]}ml^{[\frac{2q}{p}]-[\frac{q}{p}]}m\cdots l^{[\frac{kq}{p}]-[\frac{(k-1)q}{p}]}m\cdots l^{[\frac{pq}{p}]-[\frac{(p-1)q}{p}]}m \tag{5}$$

To prove this theorem, essentially the same argument as in [1] applies. The key point is to move from a torus to a punctured torus.

3. Degenerate torus surgery

Surgery on a 4-manifold M using an embedded (2-dimensional) torus T^2 in M was closely studied in [3], [4]. In this section, we will explain that our pochette surgery is nothing but a special type of a Dehn surgery along an embedded torus. Precisely speaking, a pochette surgery is a Dehn surgery along a compressibly embedded torus T^2 in the sense that a simple closed curve on T^2 bounds an embedded disk in M.

Let $ST = S^1 \times D^2$ be a 3-dimensional solid torus. Take a product $S^1 \times ST$ and attach a 2-handle H along $S^1 \times \{*\}$ with 0-framing. Then the resulting 4-manifold

$$S^1 \times ST \cup H$$

is diffeomorphic to a pochette P. The meridian m of the solid torus ST corresponds under this diffeomorphism to the meridian m on ∂P, and the longitude l of ST corresponds to the longitude l on ∂P. The circle $S^1 \times \{*\}$, $* \in \partial ST$, represents the third generator $[s]$ of $H_1(S^1 \times \partial ST; \mathbb{Z})$, which bounds a disk in P. The torus $T^2 = S^1 \times S^1 \times \{0\}$ in $S^1 \times ST \subset P$ is called the *canonical torus* of the pochette P. Let $N(T^2)$ be a thin tubular neighborhood of the canonical torus in P, which we identify with (a shrinked) $S^1 \times ST$ in the following lemma:

Lemma 7. *A diffeomorphism $h : \partial P \to \partial P$ which maps m to a curve of slope $\frac{p}{q}$ with "0-framing"is extendable to a diffeomorphism $P \setminus \mathrm{Int}N(T^2) \to P \setminus \mathrm{Int}N(T^2)$ which maps $[m]$ to $p[m] + q[l]$ on $\partial N(T^2)$. Moreover, the composite $\rho_2 h$, i.e. with "1-framing", is extendable to a diffeomorphism $P \setminus \mathrm{Int}N(T^2) \to P \setminus \mathrm{Int}N(T^2)$ which maps $[m]$ to $p[m] + q[l] + p[s]$ on $\partial N(T^2)$.*

Corollary 8. *A pochette surgery with the slope $\frac{p}{q}$ and "0-framing"is realized as a Dehn surgery on an embeded torus with the coefficient $p[m] + q[l]$, while a pochette surgery with the slope $\frac{p}{q}$ and "1-framing"is realized as a Dehn surgery with the coefficient $p[m] + q[l] + p[s]$*

Thus, the pochette surgery is an intermediate between the torus surgery and the Gluck surgery.

4. Branched coverings

There is a geometric situation which the pochette surgery naturally fits. Let Σ be a 2-knot in a 4-sphere S^4. Taking a double branched covering of S^4 branched along Σ, we obtain a closed manifold M. Suppose the projected image of Σ under the projection $p : S^4 \setminus \{\infty\} = \mathbb{R}^4 \to \mathbb{R}^3 = S^3 \setminus \{\infty\}$ has a double curve C. Problem is then "what is the change of M caused by changing the 4-dimensional heights of the two components of $p^{-1}(C) \cap \Sigma$ ".

In general, this change is described by using a torus surgery. However, in a special case where C is cointained in a 3-ball B and $B \cap p(\Sigma)$ consists of a disk D and an untwisted annulus A which meet transversely along C, the change on M is described by a pochette surgery.

O. Kataoka [5] observed this type of change as follows:

Theorem 9. (Kataoka) *(i) Suppose there exists a 4-ball B^4 in S^4 with $B^4 \cap \Sigma$ consists of an unknotted 2-disk D and an unknotted annulus A. Assume that they are separated by a properly embedded 3-disk in B^4. Then in the double branched covering M of S^4 branched along Σ, the 4-ball B^4 lifts to a pochette P.*

(ii) If another 2-knots Σ' differs from Σ only inside B^4, and $B^4 \cap \Sigma'$ consists of the unknotted 2-disk D and an unknotted annulus A' which winds around the disk D n times, then the double branched covering M' of S^4 branched along Σ' is obtained from M by removing $\mathrm{Int}P$ and pasting it back via a diffeomorphism $h : \partial P \to \partial(M \setminus \mathrm{Int}P)$ with the following property:

$$h_{\#}(m) = \begin{cases} ml^2 & \text{if } n \text{ is odd} \\ m & \text{if } n \text{ is even,} \end{cases} \qquad h_{\#}(l) = \begin{cases} l^{-1} & \text{if } n \text{ is odd} \\ l & \text{if } n \text{ is even} \end{cases}$$

where $h_{\#}$ is the induced isomorphism on the fundamental groups $F_2 \to F_2$.

As we showed in this paper, the pochette surgery has "abelian" character. This together with Kataoka's result makes it possible to identify the total spaces of the double branched coverings of S^4 branched along a number of examples of ribbon 2-knots. In particular, we have several concrete examples of ribbon 2-knots along which the double branched covering spaces of S^4 are diffeomorphic to S^4, giving counterexamples to the 4-dimensinal Smith conjecture. (Existence of such counterexamples are already known [2].)

Finally we remark that Y. Yamada [9] generalizes Kataoka's move and discusses an operation called "double ribbon-move" performed on an embedded surface in a 4-manifold M which does not change the diffeomorphism type of the total space of the double branched covering of M branched along the embedded surface.

Detailed version of the present paper will appear elsewhere.

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