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AN ENUMERATION OF THETA-CURVES WITH UP TO SEVEN CROSSINGS

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1. INTRODUCTION

In 1989, R. Litherland [4] has made a table of prime θ-curves with up to seven crossings. But the completeness of his table has not proven. We enumerate prime θ-curves with up to seven crossings by using Conway's method to confirm his table.

First we explain Conway's method. In 1969, J. H. Conway [1] has made an enumeration of prime knots and links by introducing the concept of a *tangle* and a *basic polyhedron*, defined as follows:

Definition 1.1. Let t_1 and t_2 be arcs embedded in 3-ball B^3. A pair $(B^3, t_1 \cup t_2)$ is called a *tangle* if t_1 and t_2 satisfy the following conditions (see Fig. 1):

(1) $\partial t_i = t_i \cap \partial B^3$ $(i = 1, 2)$
(2) $t_1 \cap t_2 = \emptyset$

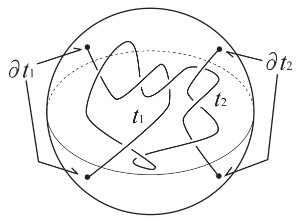


FIG. 1. Example of a tangle.

Definition 1.2. Let P be a connected 4-regular planar graph. P is called a *basic polyhedron* if it contains no bigon.

We can obtain a knot diagram from a basic polyhedron by substituting tangles for their vertices.

In Section 2, we give some definitions of a θ-curve. In Section 3, we construct prime basic polyhedra for a θ-curve. In Section 4, we give a conclusion.

2. θ-CURVE

Definition 2.1. Let Θ be a spatial graph in S^3. Θ is called a *θ-curve* if it consists of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where each edge joins two vertices.

Definition 2.2. Let Θ be a θ-curve. A *constituent knot* Θ_{ij}, $1 \le i < j \le 3$, is a subgraph of Θ that consists of two vertices (v_1, v_2) and two edges (e_i, e_j).

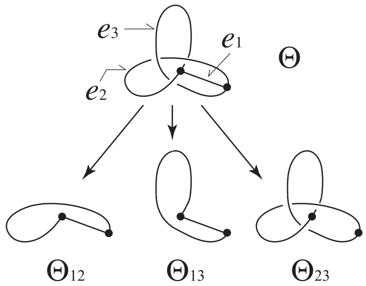


FIG. 2. Example of a θ-curve and its constituent knots.

Definition 2.3. A θ-curve is said to be *trivial* if it can be embedded in S^2.

Definition 2.4 ([4])**.** A θ-curve is said to be *prime* if it satisfies the following conditions:

(1) it is non-trivial;
(2) it is not the order-2 vertex connected sum of non-trivial knot and (possibly trivial) θ-curve;
(3) it is not the order-3 vertex connected sum of two non-trivial θ-curves.

Here an *order-n vertex connected sum* of spatial graphs is defined as in Fig. 3 (see [6]). A non-trivial knot K, such as in Fig. 3(a), is called a *local knot* of a θ-curve.

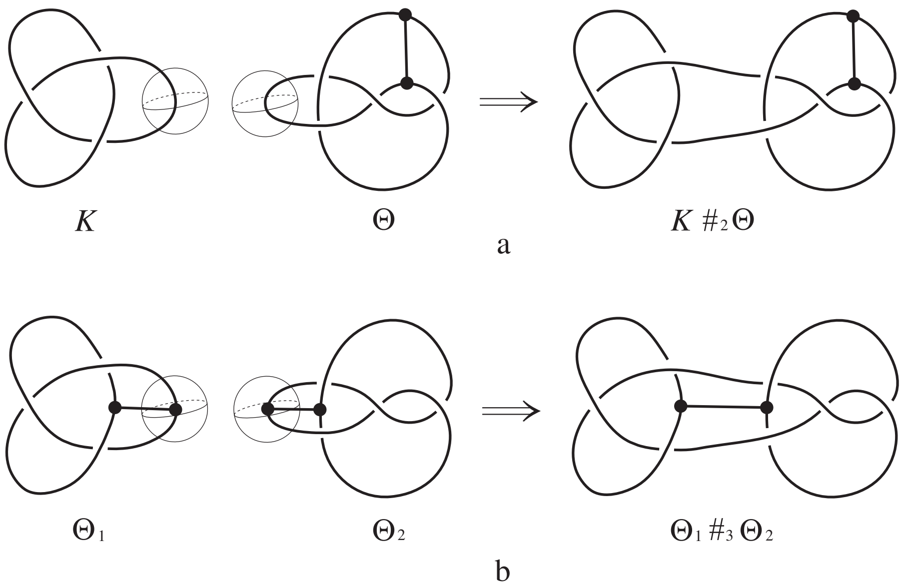


FIG. 3. (a) An order-2 vertex connected sum of spatial graphs,
(b) An order-3 vertex connected sum of spatial graphs.

3. PRIME BASIC POLYHEDRA FOR A θ-CURVE

In this section, we construct prime basic polyhedra for a θ-curve.

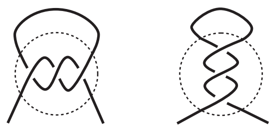
3.1. Definition of a polyhedron for a θ-curve.

Definition 3.1. Let P_Θ be a connected planar graph. P_Θ is called a *polyhedron for a θ-curve* if its two vertices are 3-valent and the other vertices are 4-valent.

We can obtain a θ-curve diagram from a polyhedron for a θ-curve by substituting algebraic tangles for their 4-valent vertices. From now on, 4-valent vertices will be denoted by $\textcircled{T}$ in the figures.

Definition 3.2. A polyhedron for a θ-curve P_Θ is said to be *basic* if it contains no loop and no bigon.

Remark 3.3. If P_Θ contains a loop, then we obtain a θ-curve diagram with a local knot or nugatory crossings from P_Θ (see Fig. 4).



a b

FIG. 4. (a) A local knot, (b) nugatory crossings.

Remark 3.4. If P_Θ contains a bigon, then we obtain a θ-curve diagram obtained from P_Θ is also obtained from another polyhedron P'_Θ with fewer 4-valent vertices than P_Θ. In fact, adding two algebraic tangles, we obtain an algebraic tangle (see Fig. 5).

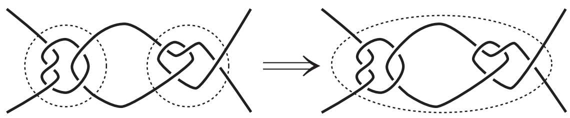
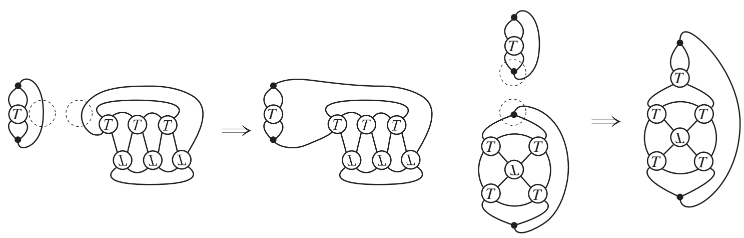


FIG. 5. The sum of algebraic tangles.

Definition 3.5. A polyhedron for a θ-curve P_Θ is said to be *prime* if it is neither order-2 nor order-3 vertex connected sum of polyhedra with fewer 4-valent vertices. Here an order-n vertex connected sum of polyhedra is defined as in Fig. 6.



a b

FIG. 6. (a) An order-2 vertex connected sum of polyhedra.
(b) An order-3 vertex connected sum of polyhedra.

Remark 3.6. An order-2 or order-3 vertex connected sum of polyhedra may produce a non-prime θ-curve.

3.2. **Prime basic 4-regular disk graphs.** In 2001, H. Yamano [8] has classified tangles of seven crossings or less in his master's thesis, where he used the concept of *prime basic* 4*-regular disk graphs*.

Definition 3.7. Let Q be a connected 4-regular planar graph, and v be a vertex of Q. We denote the neighborhood of v by $N(v)$, and the interior of $N(v)$ by $\mathring{N}(v)$. A pair (B^2, P) is called a 4*-regular disk graph* if $(B^2, P) \cong \big(S^2 \setminus \mathring{N}(v), Q \setminus \mathring{N}(v)\big)$ (see Fig. 7).

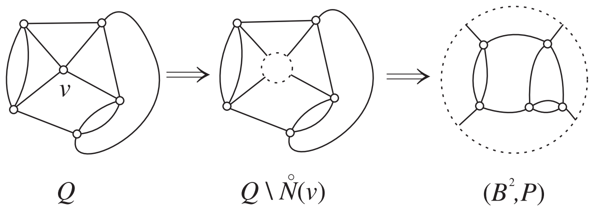


FIG. 7. The making of a 4-regular disk graph.

Definition 3.8. A 4-regular disk graph (B^2, P) is said to be *basic* if P contains no loop and no bigon.

Definition 3.9. A 4-regular disk graph (B^2, P) is said to be *prime* if for any disk D in B^2, such that ∂D meets P transversely in two points, D contains no vertex.

Yamano has obtained the following lemma:

Lemma 3.10. There exist six prime basic 4-regular disk graphs with up to seven vertices as in Fig. 8.

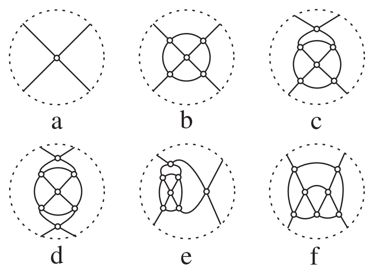


FIG. 8. (a) P_1 (b) P_5 (c) P_6
(d) P_{7-1} (e) P_{7-2} (f) P_{7-3}

3.3. **Type-$\times$ prime basic polyhedra for a θ-curve.** By using Yamano's result, we construct a prime basic polyhedron for a θ-curve whose 3-valent vertices are adjacent. We call it a *type-$\times$ prime basic polyhedon for a θ-curve*. First, we construct the numerator and the denominator of a prime basic 4-regular disk graph. Second, we add a vertex on each of the new edges. Finally we join these two vertices by an edge (see Fig. 9).

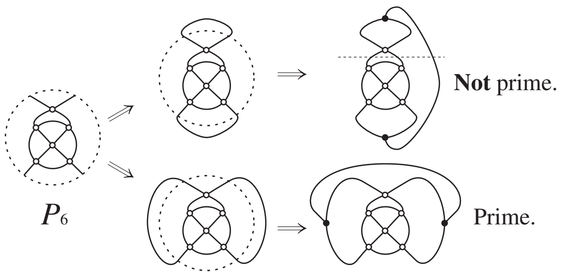


FIG. 9. The making of a polyhedron for a θ-curve.

From Lemma 3.10, we obtain the following theorem:

Theorem 3.11. There exist seven type-$\times$ prime basic polyhedra for a θ-curve with up to seven 4-valent vertices as in Fig. 10.

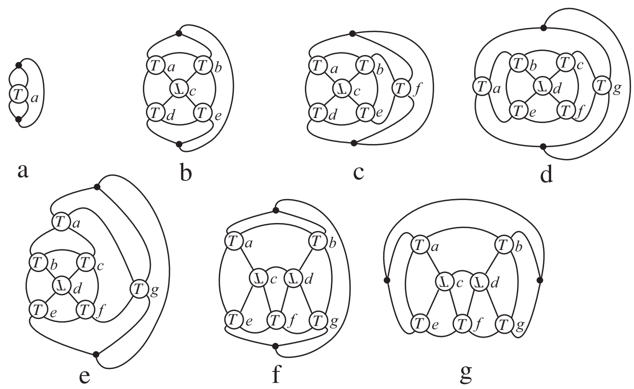


FIG. 10.
(a) $1^1_\times a$ (b) $5^1_\times a.b.c.d.e$ (c) $6^1_\times a.b.c.d.e.f$ (d) $7^1_\times a.b.c.d.e.f.g$
(e) $7^2_\times a.b.c.d.e.f.g$ (f) $7^3_\times a.b.c.d.e.f.g$ (g) $7^4_\times a.b.c.d.e.f.g$

3.4. **Type-$*$ prime basic polyhedra for a θ-curve.** In this subsection, we construct prime basic polyhedra for a θ-curve whose 3-valent vertices are **not** adjacent. We call it a *type-$*$ prime basic polyhedon for a θ-curve*. Since the proof of the following theorem is too long to give in this report, we omit it.

Theorem 3.12. There exist seventeen type-$*$ prime basic polyhedra for a θ-curve with up to seven 4-valent vertices as in Fig. 11.

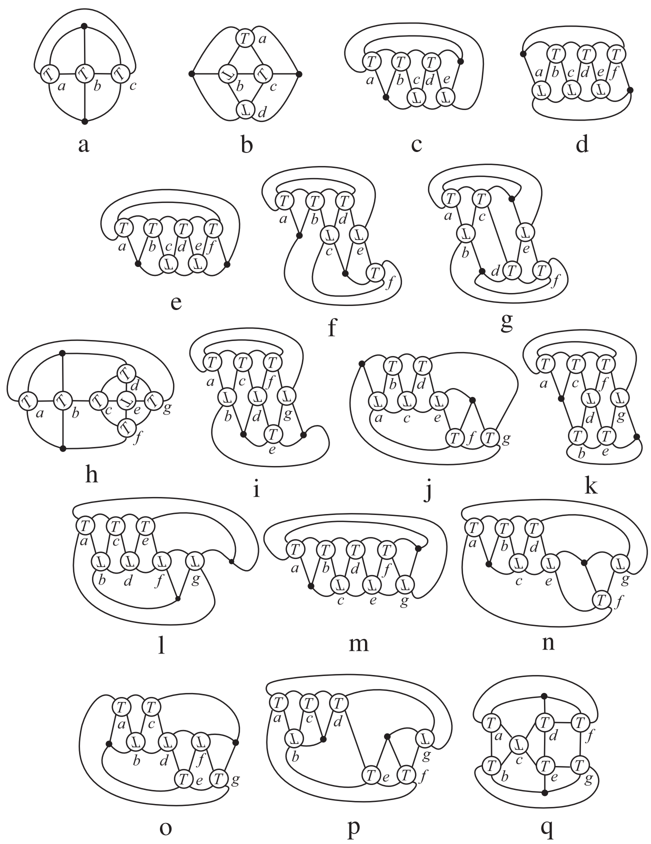


FIG. 11.
(a) $3^1_*a.b.c$ (b) $4^1_*a.b.c.d$ (c) $5^1_*a.b.c.d.e$ (d) $6^1_*a.b.c.d.e.f$
(e) $6^2_*a.b.c.d.e.f$ (f) $6^3_*a.b.c.d.e.f$ (g) $6^4_*a.b.c.d.e.f$
(h) $7^1_*a.b.c.d.e.f.g$ (i) $7^2_*a.b.c.d.e.f.g$ (j) $7^3_*a.b.c.d.e.f.g$ (k) $7^4_*a.b.c.d.e.f.g$
(l) $7^5_*a.b.c.d.e.f.g$ (m) $7^6_*a.b.c.d.e.f.g$ (n) $7^7_*a.b.c.d.e.f.g$
(o) $7^8_*a.b.c.d.e.f.g$ (p) $7^9_*a.b.c.d.e.f.g$ (q) $7^{10}_*a.b.c.d.e.f.g$

4. CONCLUSION

From Theorems 3.11 and 3.12, we can obtain all the prime θ-curves with up to seven crossings, which are the exactly same ones as in Litherland's. Litherland has shown that these θ-curves are mutually distinct by investigating the *Alexander polynomial* [3] and constituent knots.

We have shown this using the *Yamada polynomial* [7]. Furthermore, we have shown the primeness of these θ-curves by using the following proposition [4] originally due to Thurston. In fact, each θ-curve in the table has a trivial consituent knot.

Proposition 4.1. Suppose a θ-curve Θ contains a trivial constituent knot. Take the 2-fold branched cover over this trivial knot; the two lifts of the remaining edge give a knot $K^{(2)}$ in S^3. Then Θ is prime if and only if $K^{(2)}$ is prime.

APPENDIX

We give an enumeration of θ-curve with up to seven crossings by using our notation. Knots in the second column correspond to Rolfsen's knot table [5], and θ-curves in the last column correspond to Litherland's table [4]. $\bar{K}$ and $\bar{\Theta}$ denote mirror images of K and Θ, respectively.

Example 4.2. The θ-curve diagram as in Fig. 12 is denoted by $4^1_*2\ 1\ 0.1.1. -2\ 0$. Its constituent knots are $5_2, 3_1, 0_1$. Then this θ-curve and 7_{22} are ambient isotopic.

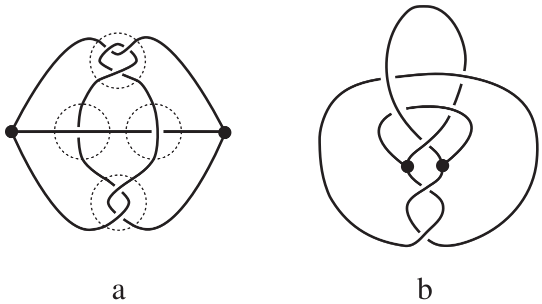


FIG. 12. (a) $4^1_*2\ 1\ 0.1.1. -2\ 0$, (b) 7_{22}.

notation	constituent knot	θ-curve
$1^1_\times\ 3$	$3_1, 0_1, 0_1$	$\bar{3}_1$
$1^1_\times\ 2\ 2$	$4_1, 0_1, 0_1$	4_1
$1^1_\times\ 5$	$5_1, 0_1, 0_1$	5_3
$1^1_\times\ 3\ 2$	$5_2, 0_1, 0_1$	5_6
$1^1_\times\ 2\ 3$	$\bar{5}_2, 0_1, 0_1$	$\bar{5}_5$
$1^1_\times\ 3, 2$	$\bar{5}_1, \bar{3}_1, 0_1$	$\bar{5}_4$
$1^1_\times\ 2\ 1, 2$	$5_2, 3_1, 0_1$	5_7
$3^1_*\ 2.2.-1$	$\bar{3}_1, 0_1, 0_1$	5_2
$4^1_*\ 2.1.1.1$	$\bar{3}_1, 0_1, 0_1$	5_2
$4^1_*\ 2\ 0.1.1.1$	$0_1, 0_1, 0_1$	5_1
$1^1_\times\ 4\ 2$	$6_1, 0_1, 0_1$	6_5
$1^1_\times\ 2\ 4$	$\bar{6}_1, 0_1, 0_1$	$\bar{6}_6$
$1^1_\times\ 3\ 1\ 2$	$6_2, 0_1, 0_1$	6_9
$1^1_\times\ 2\ 1\ 3$	$6_2, 0_1, 0_1$	6_{10}
$1^1_\times\ 2\ 1\ 1\ 2$	$6_3, 0_1, 0_1$	$\bar{6}_{14}$
$1^1_\times\ 2\ 2, 2$	$6_1, 4_1, 0_1$	6_8
$1^1_\times\ 2\ 1\ 1, 2$	$\bar{6}_2, 4_1, 0_1$	$\bar{6}_{13}$
$1^1_\times\ 3, 2\ 1$	$\bar{6}_1, 0_1, 0_1$	$\bar{6}_7$
$1^1_\times\ 3, 2+$	$\bar{6}_2, \bar{3}_1, 0_1$	$\bar{6}_{12}$
$1^1_\times\ 2\ 1, 2+$	$6_3, 3_1, 0_1$	$\bar{6}_{16}$
$3^1_*\ 3.2.-1$	$4_1, 3_1, 0_1$	6_4
$3^1_*\ 2.2.2$	$0_1, 0_1, 0_1$	5_1
$3^1_*\ 2.2.-2$	$0_1, 0_1, 0_1$	6_1
$4^1_*\ 2\ 1.1.1.1$	$0_1, 0_1, 0_1$	6_1
$4^1_*\ 2\ 1\ 0.1.1.1$	$\bar{3}_1, 0_1, 0_1$	6_2
$4^1_*\ -2\ -1\ 0.1.1.1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_4$
$4^1_*\ 2.2\ 0.1.1$	$4_1, 3_1, 0_1$	6_4
$4^1_*\ 2.1.1.2\ 0$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$4^1_*\ 2.1.1.-2\ 0$	$4_1, 3_1, 0_1$	6_4
$4^1_*\ 2\ 0.2\ 0.1.1$	$\bar{3}_1, 0_1, 0_1$	6_2
$4^1_*\ -2\ 0.2\ 0.1.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$5^1_\times\ 2.1.1.1.1$	$6_3, 0_1, 0_1$	6_{15}
$5^1_\times\ 2\ 0.1.1.1.1$	$\bar{6}_2, 0_1, 0_1$	$\bar{6}_{11}$
$5^1_*\ 2\ 0.1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$5^1_*\ 1.1.1.1.2\ 0$	$\bar{5}_2, \bar{3}_1, 0_1$	$\bar{5}_7$
$6^2_*\ 1.1.1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$6^4_*\ 1.1.1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{5}_1$

notation	constituent knot	θ-curve
$1^1_\times\ 7$	$7_1, 0_1, 0_1$	7_{25}
$1^1_\times\ 5\ 2$	$7_2, 0_1, 0_1$	$\bar{7}_{29}$
$1^1_\times\ 4\ 3$	$7_3, 0_1, 0_1$	7_{33}
$1^1_\times\ 3\ 4$	$\bar{7}_3, 0_1, 0_1$	$\bar{7}_{34}$
$1^1_\times\ 2\ 5$	$\bar{7}_2, 0_1, 0_1$	7_{28}
$1^1_\times\ 3\ 2\ 2$	$7_5, 0_1, 0_1$	$\bar{7}_{43}$
$1^1_\times\ 3\ 1\ 3$	$7_4, 0_1, 0_1$	7_{38}
$1^1_\times\ 2\ 2\ 3$	$7_5, 0_1, 0_1$	$\bar{7}_{44}$
$1^1_\times\ 2\ 2\ 1\ 2$	$7_6, 0_1, 0_1$	$\bar{7}_{53}$
$1^1_\times\ 2\ 1\ 2\ 2$	$\bar{7}_6, 0_1, 0_1$	7_{50}
$1^1_\times\ 2\ 1\ 1\ 1\ 2$	$7_7, 0_1, 0_1$	7_{59}
$1^1_\times\ 5, 2$	$\bar{7}_1, \bar{5}_1, 0_1$	$\bar{7}_{27}$
$1^1_\times\ 4\ 1, 2$	$\bar{7}_3, 5_1, 0_1$	$\bar{7}_{36}$
$1^1_\times\ 3\ 2, 2$	$7_3, \bar{5}_2, 0_1$	7_{37}
$1^1_\times\ 2\ 3, 2$	$7_2, 5_2, 0_1$	$\bar{7}_{32}$
$1^1_\times\ 3\ 1\ 1, 2$	$\bar{7}_4, 5_2, 0_1$	$\bar{7}_{42}$
$1^1_\times\ 2\ 2\ 1, 2$	$\bar{7}_5, \bar{5}_2, 0_1$	7_{49}
$1^1_\times\ (3, 2), 2$	$7_5, 5_1, 0_1$	$\bar{7}_{48}$
$1^1_\times\ (3, 2), -2$	$5_1, 5_2, 0_1$	7_{18}
$1^1_\times\ (3, -2), 2$	$6_2, 0_1, 0_1$	6_{11}
$1^1_\times\ (3, -2), -2$	$6_3, 0_1, 0_1$	6_{15}
$1^1_\times\ (2\ 1, 2), 2$	$\bar{7}_6, \bar{5}_2, 0_1$	7_{58}
$1^1_\times\ (2\ 1, 2), -2$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
$1^1_\times\ (2\ 1, -2), 2$	$6_3, 0_1, 0_1$	$\bar{6}_{15}$
$1^1_\times\ (2\ 1, -2), -2$	$\bar{6}_2, 0_1, 0_1$	$\bar{6}_{11}$
$1^1_\times\ 4, 3$	$\bar{7}_1, \bar{3}_1, 0_1$	$\bar{7}_{26}$
$1^1_\times\ 3\ 1, 3$	$7_3, \bar{3}_1, 0_1$	7_{35}
$1^1_\times\ 2\ 2, 3$	$7_2, 0_1, 0_1$	$\bar{7}_{30}$
$1^1_\times\ 4, 2\ 1$	$7_2, 3_1, 0_1$	$\bar{7}_{31}$
$1^1_\times\ 3\ 1, 2\ 1$	$7_5, 3_1, 0_1$	$\bar{7}_{45}$
$1^1_\times\ 2\ 1\ 1, 2\ 1$	$\bar{7}_6, 0_1, 0_1$	7_{54}
$1^1_\times\ 2\ 2, 2+$	$\bar{7}_6, 4_1, 0_1$	7_{57}
$1^1_\times\ 2\ 2, -2-$	$6_2, 4_1, 0_1$	6_{13}
$1^1_\times\ 2\ 1\ 1, 2+$	$\bar{7}_7, 4_1, 0_1$	$\bar{7}_{65}$
$1^1_\times\ 2\ 1\ 1, -2-$	$\bar{6}_1, 4_1, 0_1$	$\bar{6}_8$
$1^1_\times\ 3, 3+$	$\bar{7}_4, 0_1, 0_1$	$\bar{7}_{39}$
$1^1_\times\ 3, -3-$	$\bar{6}_1, 0_1, 0_1$	$\bar{6}_7$

notation	constituent knot	θ-curve
$1^1_\times$ $2\,1, 2\,1+$	$7_7, 0_1, 0_1$	7_{62}
$1^1_\times$ $2\,1, 2\,1-$	$\bar{3}_1, 0_1, 0_1$	5_2
$1^1_\times$ $2\,1, -2-1+$	$6_1, 0_1, 0_1$	6_7
$1^1_\times$ $2\,1, -2-1-$	$\bar{6}_1, 0_1, 0_1$	$\bar{6}_7$
$1^1_\times$ $3, 2++$	$\bar{7}_5, \bar{3}_1, 0_1$	7_{46}
$1^1_\times$ $3, -2--$	$6_3, \bar{3}_1, 0_1$	6_{16}
$1^1_\times$ $2\,1, 2++$	$7_6, 3_1, 0_1$	$\bar{7}_{56}$
$1^1_\times$ $2\,1, -2--$	$6_2, 3_1, 0_1$	6_{12}
3^1_* $(3,2).1.-1$	$\bar{6}_2, 0_1, 0_1$	$\bar{6}_{11}$
3^1_* $(3,-2).1.-1$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
3^1_* $(2\,1,2).1.-1$	$6_3, 0_1, 0_1$	6_{15}
3^1_* $(2\,1,-2).1.-1$	$5_1, 5_2, 0_1$	7_{18}
3^1_* $4.2.-1$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{20}$
3^1_* $3\,1.2.-1$	$\bar{5}_1, 0_1, 0_1$	7_{15}
3^1_* $2\,1\,1.2.-1$	$5_2, 3_1, 0_1$	7_{22}
3^1_* $2\,1\,1\,0.-2.1$	$\bar{6}_2, \bar{3}_1, 0_1$	$\bar{6}_{12}$
3^1_* $3.2\,1.-1$	$5_2, 4_1, 0_1$	$\bar{7}_{24}$
3^1_* $2\,1.2\,1.-1$	$\bar{5}_1, 0_1, 0_1$	7_{17}
3^1_* $3.2.2$	$\bar{3}_1, 0_1, 0_1$	6_2
3^1_* $3.2.-2$	$4_1, 0_1, 0_1$	$\bar{7}_{13}$
3^1_* $3.-2.2$	$4_1, 0_1, 0_1$	$\bar{7}_{13}$
3^1_* $3.-2.-2$	$3_1, 0_1, 0_1$	$\bar{7}_6$
3^1_* $3\,0.-2.2$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_4$
3^1_* $3\,0.-2.-2$	$3_1, \bar{3}_1, 0_1$	7_8
3^1_* $2\,1.-2\,0.2\,0$	$\bar{3}_1, 0_1, 0_1$	6_2
3^1_* $2\,1.-2\,0.-2\,0$	$\bar{3}_1, 0_1, 0_1$	7_7
4^1_* $4.1.1.1$	$3_1, \bar{3}_1, 0_1$	$\bar{7}_8$
4^1_* $4\,0.1.1.1$	$\bar{3}_1, 0_1, 0_1$	7_5
4^1_* $-4\,0.1.1.1$	$5_1, 0_1, 0_1$	$\bar{7}_{15}$
4^1_* $3\,1.1.1.1$	$\bar{3}_1, 0_1, 0_1$	7_6
4^1_* $3\,1\,0.1.1.1$	$0_1, 0_1, 0_1$	$\bar{7}_1$
4^1_* $-3-1\,0.1.1.1$	$5_2, 0_1, 0_1$	7_{20}
4^1_* $2\,2.1.1.1$	$\bar{3}_1, 0_1, 0_1$	7_7
4^1_* $2\,2\,0.1.1.1$	$3_1, \bar{3}_1, 0_1$	7_{10}
4^1_* $-2-2\,0.1.1.1$	$\bar{5}_2, \bar{3}_1, 0_1$	$\bar{7}_{23}$
4^1_* $2\,1.2\,0.1.1$	$4_1, 0_1, 0_1$	$\bar{7}_{13}$
4^1_* $2\,1\,0.2\,0.1.1$	$4_1, 4_1, 0_1$	7_{14}
4^1_* $-2-1\,0.2\,0.1.1$	$4_1, 0_1, 0_1$	7_{13}

notation	constituent knot	θ-curve
$4^1_*\ 3.1.1.2$	$5_1, 0_1, 0_1$	7_{16}
$4^1_*\ 3\ 0.1.1.2$	$5_2, 4_1, 0_1$	7_{23}
$4^1_*\ -3\ 0.1.1.2$	$\bar{5}_2, 4_1, 0_1$	$\bar{7}_{24}$
$4^1_*\ 2\ 1.1.1.2$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{21}$
$4^1_*\ 3.1.1.2\ 0$	$4_1, 0_1, 0_1$	7_{11}
$4^1_*\ 3.1.1.-2\ 0$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{20}$
$4^1_*\ 3\ 0.1.1.2\ 0$	$0_1, 0_1, 0_1$	$\bar{7}_2$
$4^1_*\ 3\ 0.1.1.-2\ 0$	$\bar{5}_1, 0_1, 0_1$	7_{15}
$4^1_*\ -3\ 0.1.1.2\ 0$	$5_1, 0_1, 0_1$	$\bar{7}_{17}$
$4^1_*\ 2\ 1\ 0.1.1.2\ 0$	$4_1, 0_1, 0_1$	7_{12}
$4^1_*\ 2\ 1\ 0.1.1.-2\ 0$	$5_2, 3_1, 0_1$	7_{22}
$4^1_*\ -2\ -1\ 0.1.1.2\ 0$	$\bar{5}_2, 4_1, 0_1$	$\bar{7}_{24}$
$4^1_*\ 2.3.1.1$	$\bar{5}_1, 0_1, 0_1$	7_{15}
$4^1_*\ 2.3\ 0.1.1$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{20}$
$4^1_*\ 2.2\ 1.1.1$	$5_2, 3_1, 0_1$	7_{22}
$4^1_*\ 2\ 0.3.1.1$	$\bar{3}_1, 0_1, 0_1$	7_5
$4^1_*\ -2\ 0.3.1.1$	$3_1, \bar{3}_1, 0_1$	7_8
$4^1_*\ 2\ 0.3\ 0.1.1$	$0_1, 0_1, 0_1$	$\bar{7}_1$
$4^1_*\ -2\ 0.3\ 0.1.1$	$3_1, 0_1, 0_1$	$\bar{7}_6$
$4^1_*\ 2\ 0.2\ 1.1.1$	$3_1, \bar{3}_1, 0_1$	7_{10}
$4^1_*\ -2\ 0.2\ 1.1.1$	$3_1, 0_1, 0_1$	$\bar{7}_7$
$4^1_*\ 2.2.2.1$	$5_2, 4_1, 0_1$	7_{24}
$4^1_*\ 2.2.2\ 0.1$	$\bar{5}_1, 0_1, 0_1$	7_{17}
$4^1_*\ -2.2.2\ 0.1$	$\bar{5}_1, 0_1, 0_1$	7_{17}
$4^1_*\ 2.2.1.2$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
$4^1_*\ 2.2.1.2\ 0$	$5_2, 0_1, 0_1$	7_{19}
$4^1_*\ 2.2.1.-2\ 0$	$4_1, 0_1, 0_1$	7_{13}
$4^1_*\ 2\ 0.2\ 0.2.1$	$4_1, 4_1, 0_1$	7_{14}
$4^1_*\ -2\ 0.2\ 0.2.1$	$\bar{3}_1, \bar{3}_1, 0_1$	7_9

notation	constituent knot	θ-curve
$5^1_\times$ 2 1.1.1.1.1	$\bar{7}_7, 0_1, 0_1$	$\bar{7}_{63}$
$5^1_\times$ $-2-1.1.1.1.1$	$\bar{6}_1, 4_1, 0_1$	$\bar{6}_8$
$5^1_\times$ 2 1 0.1.1.1.1	$\bar{7}_6, 0_1, 0_1$	$\bar{7}_{51}$
$5^1_\times$ 2.1.1.2.1	$7_7, 0_1, 0_1$	7_{60}
$5^1_\times$ $2.1.1.-2.1$	$6_2, 0_1, 0_1$	6_{11}
$5^1_\times$ $-2.1.1.-2.1$	$5_1, 0_1, 0_1$	7_{16}
$5^1_\times$ 2.1.1.1.2	$7_7, \bar{3}_1, 0_1$	7_{64}
$5^1_\times$ $2.1.1.1.-2$	$6_2, 4_1, 0_1$	6_{13}
$5^1_\times$ 2.2 0.1.1.1	$\bar{7}_6, 0_1, 0_1$	7_{52}
$5^1_\times$ $2.-2$ 0.1.1.1	$\bar{5}_1, 0_1, 0_1$	$\bar{7}_{16}$
$5^1_\times$ -2.2 0.1.1.1	$6_3, 0_1, 0_1$	$\bar{6}_{15}$
$5^1_\times$ $-2.-2$ 0.1.1.1	$\bar{6}_2, 0_1, 0_1$	$\bar{6}_{11}$
$5^1_\times$ 2.1.2 0.1.1	$7_6, 3_1, 0_1$	$\bar{7}_{55}$
$5^1_\times$ 2 0.1.2 0.1.1	$\bar{7}_5, \bar{3}_1, 0_1$	7_{47}
$5^1_\times$ 2 0.1.1.2 0.1	$\bar{7}_4, 3_1, 0_1$	$\bar{7}_{41}$
$5^1_\times$ 2 0.1.1.1.2 0	$\bar{7}_4, 3_1, 0_1$	$\bar{7}_{40}$
5^1_* $-1.-3.1.1.1$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
5^1_* $1.1.1.1.-3$	$5_1, 5_2, 0_1$	7_{18}
5^1_* 3 0.1.1.1.1	$4_1, 0_1, 0_1$	7_{13}
5^1_* -3 0.1.1.1.1	$4_1, 0_1, 0_1$	$\bar{7}_{13}$
5^1_* -1.3 0.1.1.1	$6_1, 4_1, 0_1$	6_8
5^1_* 1.1.1.1.3 0	$\bar{6}_1, 4_1, 0_1$	$\bar{6}_8$
5^1_* -1.2 1.1.1.1	$6_3, 0_1, 0_1$	$\bar{6}_{15}$
5^1_* $1.1.1.1.-2-1$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
5^1_* $-2.-2.1.1.1$	$\bar{5}_1, 0_1, 0_1$	$\bar{7}_{16}$
5^1_* $-2.-1.-1.-1.2$	$\bar{5}_1, 0_1, 0_1$	$\bar{7}_{16}$
5^1_* $-1.-2.1.2.1$	$5_1, 0_1, 0_1$	7_{16}
5^1_* $-1.-2.1.1.2$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
5^1_* $-1.-2.-1.-1.2$	$\bar{5}_1, \bar{5}_2, 0_1$	$\bar{7}_{18}$
5^1_* $-1.-1.-2.-1.2$	$5_1, 0_1, 0_1$	7_{16}
5^1_* -2.2 0.1.1.1	$4_1, 3_1, 0_1$	6_3
5^1_* 2.1.1.2 0	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
5^1_* $-1.-2.1.2$ 0.1	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{21}$
5^1_* $-1.-2.1.-2$ 0.1	$4_1, 3_1, 0_1$	6_3
5^1_* 1.2.1.1.2 0	$6_2, 4_1, 0_1$	6_{13}
5^1_* $-1.-2.1.1.2$ 0	$5_1, 5_2, 0_1$	7_{18}
5^1_* 1.1.1.2.2 0	$\bar{6}_1, 0_1, 0_1$	$\bar{6}_7$

notation	constituent knot	θ-curve
$5^1_*\ 2\ 0.2.1.1.1$	$\bar{3}_1, \bar{3}_1, 0_1$	7_9
$5^1_*\ 2\ 0.-2.1.1.1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$5^1_*\ -2\ 0.2.1.1.1$	$0_1, 0_1, 0_1$	7_4
$5^1_*\ -2\ 0.-2.1.1.1$	$5_2, 0_1, 0_1$	7_{21}
$5^1_*\ 2\ 0.1.2.1.1$	$4_1, 0_1, 0_1$	7_{13}
$5^1_*\ -2\ 0.1.2.1.1$	$\bar{3}_1, 0_1, 0_1$	7_6
$5^1_*\ 2\ 0.1.1.2.1$	$3_1, 0_1, 0_1$	$\bar{7}_6$
$5^1_*\ -2\ 0.1.1.2.1$	$4_1, 0_1, 0_1$	$\bar{7}_{13}$
$5^1_*\ 2\ 0.1.1.1.2$	$0_1, 0_1, 0_1$	$\bar{7}_4$
$5^1_*\ 2\ 0.1.1.1.-2$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{21}$
$5^1_*\ -2\ 0.1.1.1.2$	$3_1, 3_1, 0_1$	$\bar{7}_9$
$5^1_*\ -2\ 0.1.1.1.-2$	$4_1, 3_1, 0_1$	6_3
$5^1_*\ -1.2\ 0.2.1.1$	$6_1, 0_1, 0_1$	6_7
$5^1_*\ -1.2\ 0.1.1.2$	$\bar{6}_2, 4_1, 0_1$	$\bar{6}_{13}$
$5^1_*\ -1.-2\ 0.-1.-1.2$	$5_1, 5_2, 0_1$	7_{18}
$5^1_*\ -1.-1.2\ 0.-1.2$	$4_1, 3_1, 0_1$	6_3
$5^1_*\ -1.-1.-2\ 0.-1.2$	$\bar{5}_2, 0_1, 0_1$	$\bar{7}_{21}$
$5^1_*\ -1.2\ 0.1.2\ 0.1$	$\bar{6}_2, 0_1, 0_1$	$\bar{6}_{11}$
$5^1_*\ -1.2\ 0.1.-2\ 0.1$	$\bar{5}_1, 0_1, 0_1$	$\bar{7}_{16}$
$5^1_*\ -1.-1.2\ 0.2\ 0.1$	$0_1, 0_1, 0_1$	6_1
$5^1_*\ -1.-1.2\ 0.-2\ 0.1$	$3_1, 0_1, 0_1$	$\bar{6}_2$
$5^1_*\ -1.-1.-2\ 0.2\ 0.1$	$\bar{5}_1, 0_1, 0_1$	$\bar{7}_{16}$
$5^1_*\ -1.-1.-2\ 0.-2\ 0.1$	$0_1, 0_1, 0_1$	6_1
$5^1_*\ 1.1.2\ 0.1.2\ 0$	$6_2, 0_1, 0_1$	6_{11}
$5^1_*\ 1.1.-2\ 0.1.2\ 0$	$5_1, 0_1, 0_1$	7_{16}
$6^1_*\ 1.2.1.1.1.1$	$0_1, 0_1, 0_1$	7_2
$6^1_*\ 1.2\ 0.1.1.1.1$	$4_1, 0_1, 0_1$	7_{12}
$6^1_*\ 1.-2\ 0.1.1.1.1$	$4_1, 3_1, 0_1$	6_3
$6^2_*\ 1.1.1.2.1.1$	$3_1, 0_1, 0_1$	$\bar{7}_6$
$6^2_*\ 2\ 0.1.1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{7}_3$
$6^2_*\ -2\ 0.1.1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{7}_4$
$6^2_*\ 1.2\ 0.1.1.1.1$	$4_1, 0_1, 0_1$	7_{13}
$6^2_*\ 1.1.2\ 0.1.1.1$	$\bar{3}_1, \bar{3}_1, 0_1$	7_9
$6^2_*\ 1.1.-2\ 0.1.1.1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$

notation	constituent knot	θ-curve
$6^3_*\,2.1.-1.1.1.1$	$3_1, 3_1, 0_1$	$\bar{3}_1\#_3\bar{3}_1$
$6^3_*\,2.1.1.1.-1.-1$	$4_1, 3_1, 0_1$	6_4
$6^3_*\,1.1.2.1.1.1$	$6_3, 0_1, 0_1$	6_{15}
$6^3_*\,1.1.-2.1.1.1$	$5_2, 0_1, 0_1$	7_{21}
$6^3_*\,1.1.2.1.-1.-1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$6^3_*\,1.1.-2.1.-1.-1$	$5_1, 0_1, 0_1$	7_{16}
$6^3_*\,1.1.1.1.2.-1$	$6_3, 0_1, 0_1$	6_{15}
$6^3_*\,1.1.1.1.-2.-1$	$5_2, 0_1, 0_1$	7_{21}
$6^3_*\,1.1.-1.1.2.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$6^3_*\,1.1.-1.1.-2.1$	$5_1, 0_1, 0_1$	7_{16}
$6^3_*\,2\,0.1.1.1.-1.-1$	$\bar{3}_1, 0_1, 0_1$	6_2
$6^3_*\,2\,0.1.-1.1.1.1$	$3_1, \bar{3}_1, 0_1$	$3_1\#_3\bar{3}_1$
$6^3_*\,2.-1.1.-1.-1.-1$	$\bar{3}_1, 0_1, 0_1$	3_1
$6^3_*\,2.-1.-1.-1.1.1$	$0_1, 0_1, 0_1$	6_1
$6^3_*\,1.2\,0.1.1.-1.-1$	$3_1, \bar{3}_1, 0_1$	$3_1\#_3\bar{3}_1$
$6^3_*\,1.2\,0.-1.1.1.1$	$\bar{3}_1, 0_1, 0_1$	6_2
$6^3_*\,1.-2\,0.1.1.-1.-1$	$3_1, 0_1, 0_1$	$\bar{3}_1$
$6^3_*\,1.-2\,0.-1.1.1.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$6^3_*\,1.1.2\,0.1.-1.-1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$6^3_*\,1.1.-2\,0.1.1.1$	$0_1, 0_1, 0_1$	6_1
$6^3_*\,1.1.1.1.-2\,0.-1$	$0_1, 0_1, 0_1$	6_1
$6^3_*\,1.1.-1.1.-2\,0.1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$6^4_*\,2\,0.1.1.1.1.1$	$3_1, 0_1, 0_1$	$\bar{6}_2$
$6^4_*\,1.2\,0.1.1.1.1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$6^4_*\,1.1.2\,0.1.1.1$	$0_1, 0_1, 0_1$	$\bar{6}_1$
$7^4_\times\,1.1.1.1.1.1.1$	$7_7, 0_1, 0_1$	7_{61}
$7^3_*\,1.1.1.1.1.1.1$	$6_3, 0_1, 0_1$	$\bar{6}_{15}$
$7^4_*\,1.1.1.1.1.1.1$	$0_1, 0_1, 0_1$	7_4
$7^4_*\,1.-1.1.-1.-1.1.-1$	$4_1, \bar{3}_1, 0_1$	$\bar{6}_3$
$7^9_*\,1.1.1.1.1.1.1$	$\bar{3}_1, 0_1, 0_1$	6_2

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