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A COMBINATORIAL CALCULUS FOR $\mathcal{A}$-SPACES

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In loving memory of our master and teacher Rabbi Binyomin Ze'ev Kahane Hy"d

Abstract. We sketch a direct combinatorial method for analyzing Jacobi diagrammes. This method suggests a new algebraic and homological-algebraic structure on such spaces in all degrees, and for low loop degrees provides us with an algorithm to explicitly (recursively) calculate their dimensions.

1. Introduction

Jacobi diagrammes are basic, elegant indexing objects in the theory of quantum invariants. They are related to all kinds of good things, like the Associator, a deep and mysterious object first described by Drinfel'd which seems to pop up all over quantum mathematics; and the universal finite-type knot or link invariant, the Kontsevich Integral, which we could calculate if only we could find a rational Associator.

Knowing about our indexing set can tell us a lot about the objects they index. It can tell us how many there are, for example, and so trying to enumerate Jacobi diagrammes has been a popular way to attempt to ascertain at least approximately how many Vassiliev invariants there are of every type.

First, let us recall the definition of a Jacobi diagramme ([4],[6]).

Definition 1.1. A *skeleton* is a finite ordered collection of components, which are either oriented intervals, oriented circles, or colours.

Definition 1.2. A *Jacobi diagramme based on a skeleton M* is a finite graph D made of 1-dimensional components of M and undirected arcs, with two types of vertices allowed:

(1) *External* vertices which end in a component of M. Arcs ending in external vertices are called *legs*.
(2) *Internal* vertices in which three arcs meet. These vertices are oriented— one of the two possible cyclic orderings of the arcs meeting in such a vertex is specified.

Remark 1.3. Here we assume all Jacobi diagrammes to have a finite number of vertices.

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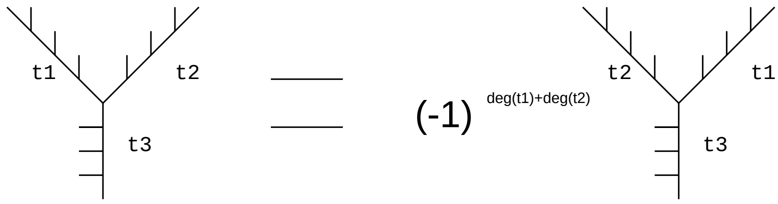

FIGURE 1. The AS relation on trees

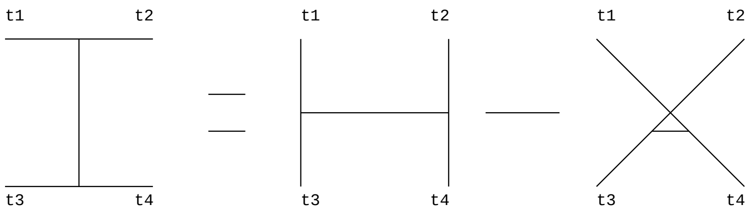

FIGURE 2. The IHX relation on trees

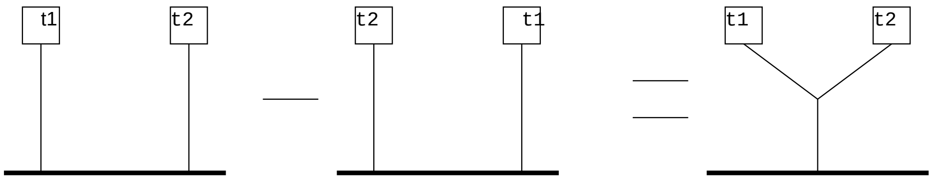

FIGURE 3. The STU relation on disjoint trees. The thick line is a component of the skeleton.

The collection of all Jacobi diagrammes based on a skeleton M will be denoted $\mathcal{D}(M)$. Its sub-collection consisting of diagrammes having at least one external vertex in each connected component will be denoted by $\mathcal{D}'(M)$. The sub-collection of $\mathcal{D}'(M)$ consisting of diagrammes which are connected graphs will be denoted $\mathcal{D}'_c(M)$.

Definition 1.4.

$$\mathcal{A}(M) := \mathrm{span}_{\Bbbk}(\mathcal{D}(M))/\{AS,\ IHX,\ \text{and}\ STU\ \text{relations}\}$$

$\mathcal{A}'(M)$ and $\mathcal{A}'_c(M)$ are analogously defined as quotients of $\mathrm{span}_{\Bbbk}(\mathcal{D}'(M))$ and of $\mathrm{span}_{\Bbbk}(\mathcal{D}'_c(M))$ respectively. Here, $\Bbbk$ is the field we are working of, which we assume not to be of characteristic 2. $\mathcal{A}(M)$ is called an $\mathcal{A}-$*Space.*

Remark 1.5. Clearly every diagramme in an $\mathcal{A}$-space is in particular a Jacobi diagramme.

For M the disjoint union of p oriented intervals, $\mathcal{A}(M)$, $\mathcal{A}'(M)$, and $\mathcal{A}'_c(M)$ shall be called simply $\mathcal{A}(p)$, $\mathcal{A}'(p)$, and $\mathcal{A}'_c(p)$.

The spaces of Jacobi diagrammes and $\mathcal{A}$-spaces defined above admit two natural filtrations. The first is to filter by half the total number of vertices on each diagramme— this we call the *degree filtration.* The other option is to filter by half the difference between the number of internal vertices and the number of legs. Roughly speaking this gives us the number of loops in a diagramme, and it is appropriately called the *loop degree filtration.* Which degree and loop-degree a Vassiliev invariant appears with in the Kontsevich Integral can tell us quite a lot about it. The degree gives us its type, and the loop degree seems to tell us roughly what the Vassiliev invariant in question is measuring ([3]).

It will be useful to introduce the following notation.

Notation. For any $a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq -1$

$$\mathcal{A}'(p)^a := \mathcal{A}'(p)/\text{span}\{D \in \mathcal{A}'(p) \mid deg(D) > a\}$$

$$\mathcal{A}'(p)_b := \mathcal{A}'(p)/\text{span}\{D \in \mathcal{A}'(p) \mid ld(D) > b\}$$

$$\mathcal{A}'(p)^a_b := \mathcal{A}'(p)^a \cap \mathcal{A}'(p)_b$$

$$\mathcal{A}'_c(p)^a := \mathcal{A}'_c(p)/\text{span}\{D \in \mathcal{A}'_c(p) \mid deg(D) > a\}$$

$$\mathcal{A}'_c(p)_b := \mathcal{A}'_c(p)/\text{span}\{D \in \mathcal{A}'_c(p) \mid ld(D) > b\}$$

$$\mathcal{A}'_c(p)^a_b := \mathcal{A}'_c(p)^a \cap \mathcal{A}'_c(p)_b$$

Where we are dealing with spaces of Jacobi diagrammes rather than $\mathcal{A}$-spaces, the letter $\mathcal{D}$ shall replace the letter $\mathcal{A}$.

What I would like to do is to sketch a combinatorial algebraic method which I would like to utilize in order to analyze these spaces. For me, this was a great voyage of rediscovery— when I did it I thought it was new, but most of the results turned out to be not as new as I had thought, at least for the lowest loop degrees. In particular, the isomorphism between Jacobi diagrammes of loop degree -1 and a certain algebra related to free Lie algebras turned out to be something that is well known among experts, as Kazuo Habiro pointed out to me.

In non-positive loop degrees, we can give the dimension of our space of invariants and calculate its basis (or at least we can find an efficient recursive algorithm to do so). And the technique provides us a window into understanding the world of higher loop-degree $\mathcal{A}$-spaces as well.

2. Loop Degree -1

To say that the loop degree of a Jacobi diagramme is -1 is basically the same as to say that as a graph, it contains no non-trivial cycles. To put it another way, a loop degree -1 Jacobi diagramme is what graph theorists would call a *tree.*

In $\mathcal{A}'_c(p)_{-1}$, an STU move is just the identity, because it would add a loop, and so the only moves we have to take into account are AS and IHX. I'm going to want to use a meta-move, which combines these two operations, which we shall call *breaking a tree into branches* as in Figure 4.

My initial idea for trees was to first take them all into a standard form, which turns out to be called a *comb.* This is a tree where all trivelant vertices lie on a single non-backtracking path in the graph. See Figure 5.

I proved that there is a well-defined map from the algebra of trees to the algebra of combs, which has 2 stages.

Stage 1: Choose two leaves as “heads” as in Figure 6.
Stage 2: We may break down the trees $T_1, T_2, \ldots, T_m$ into leaves, as in Figure 4.

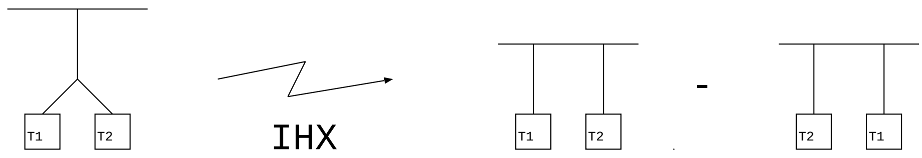

FIGURE 4. Breaking a tree into branches

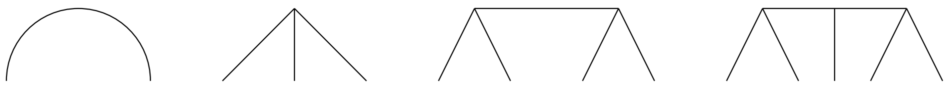

FIGURE 5. Examples of combs of degree ≤ 5 (skeleton is ignored)

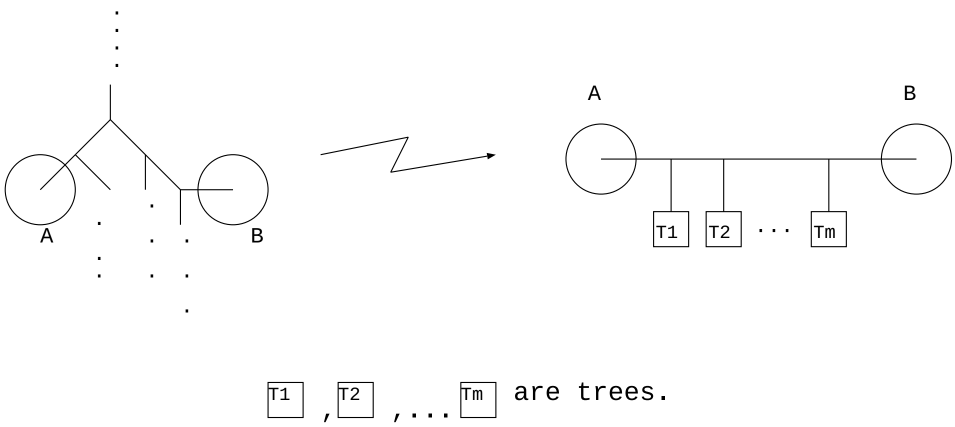

FIGURE 6. Choosing two heads, and laying out the graph according to them

We then show that any two choices made in stage 1 lead to the same comb after "fold moves" as in Figure 7.

But trying to find a basis for this space turned out to be a difficult task, because we can "fold" both ends. This leads us to the idea of fixing one end, and folding only one side first. So now instead of looking at trees, we are looking at *rooted trees*.

Rooted trees turn out to be quite prevalent structures in the fields of Mathematics and of Computer Science. It turns out that they have a very natural algebraic interpretation, via the idea of a *magma*, which is simply a set with a binary operation. A *free magma* $M(A)$ over a set A can be associated with a rooted tree.

But what we have here is not simply rooted trees— it is rooted trees modulo the AS and IHX relations. AS, for a magma, says that $xy = -yx$ for all x and y elements in $M(A)$, and IHX gives us $(xy)z + x(yz) + (zx)y = 0$. These relations are known in algebra as *antisymmetry* and the *Jacobi identity*- the two defining relations of a free Lie algebra!

So it turns out that the algebra of rooted trees as a subalgebra of $M(A)$ modulo AS and IHX is isomorphic to the free Lie algebra on A, which we shall denote $\mathcal{L}(A)$ following Reutenauer ([10]).

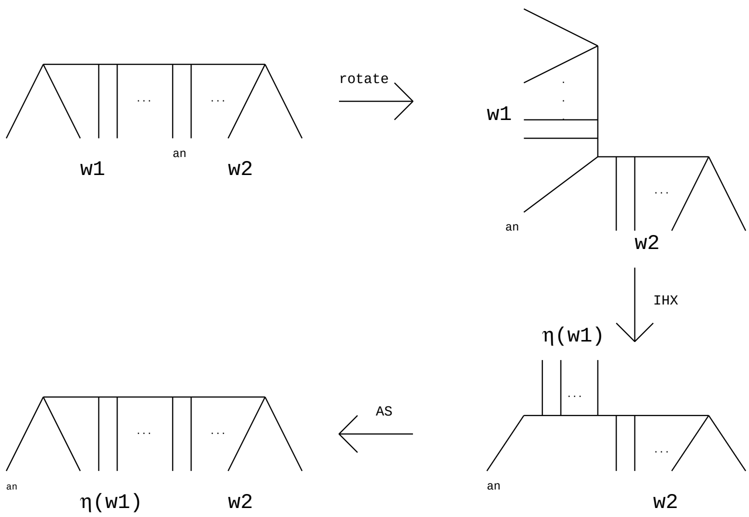

FIGURE 7. "Fold moves" on combs

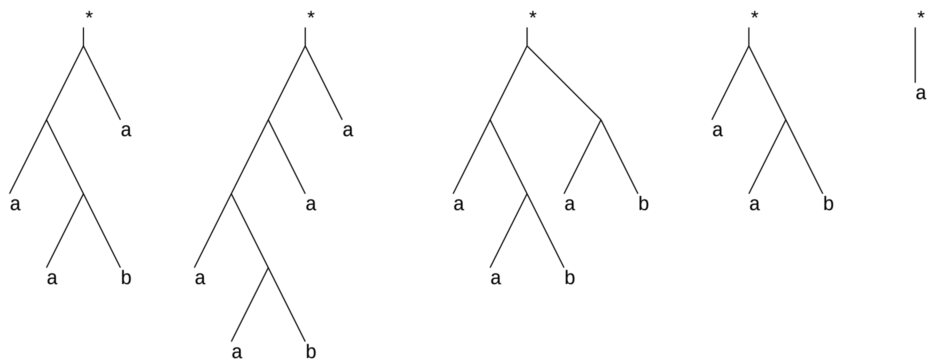

FIGURE 8. The first five trees of degree ≤ 5 of a Hall set in M(A)

The theory of free Lie algebras is a large and beautiful subject which dates back to the turn of the century. We know all kinds of nice things about them— we know their order, and we can find explicit bases for them.

Definition 2.1 ([10], Theorem 5.8)**.** A Hall word h over A is an element of $M(A)$ in which for each nontrivial factorization $h = uv$, one has $h < v$ (see Figure 8). A Hall set is the set of all Hall words.

Remark 2.2. In Reutenauer's book, a different definition is given, and it is a nontrivial theorem that the definition he gives is equivalent to the definition given above.

Theorem 2.3 ([10], Theorem 4.9(i))**.** *The Hall set over A is a basis for the free Lie algebra $\mathcal{L}(A)$.*

Corollary 2.4 ([10], Corollary 4.14)**.** *The number of words of length n in the free Lie algebra on p generators is*

$$\frac{1}{n}\sum_{d|n}\mu(d)p^{\frac{n}{d}}$$

where μ is the Möbius function.

Now that we know the relevant properties of free Lie algebras, we may use them to deduce properties of the space we are interested in, $\mathcal{A}'_c(p)_{-1}$, via the exact sequence

$$\mathbf{h}(\underline{p})_n \longrightarrow \mathcal{L}_n(\underline{p-1}) \longrightarrow \mathcal{L}_{n-1}(\underline{p}) \longrightarrow 0 \tag{2.1}$$

Where $\mathbf{h}(\underline{p})_n$ is the subalgebra of $F_{\Bbbk}(\underline{p})$ (the free $\Bbbk$-algebra on p generators) which is isomorphic to $\mathcal{A}'_c(p)^n_{-1}$. In particular we may use this to calculate the degree of $\mathcal{A}'_c(p)_{-1}$.

Remark 2.5. In my talk in Korea, the free Lie algebras in the above exact sequence were the wrong way around. The correct SES is the one given above. I thank Fred Cohen for pointing out this error in my talk.

Remark 2.6. $F_{\Bbbk}(-)$, $\mathcal{L}(-)$, and $\mathbf{h}(-)$ are all functors from **FinSet** to the category of $\Bbbk$-algebras of finite words **FinWord**. By combining with the forgetful functor we may iterate these functors, as we shall do later on.

In order to find $\mathcal{A}'_c(p)_{-1}$ explicitly we need a new idea. Let us pick one of the elements of the skeleton, i, and think of legs attached to it as 'very small'. We now take the tree that is our original tree but with the 'very small legs' erased, and relabel each leg by the pair (j, a) where j is the original label and a is the number of small legs on the original tree that lie between it and the next trivelant vertex. We may assume without loss of generality by IHX that there are no 'very small legs' between two trivelant vertices in this new tree.

In this way, we have reduced the degree of the tree which we are looking at, and this therefore allows a recursive calculation of $\mathcal{A}'_c(p)^n_{-1}$. The algorithm is made more efficient because for many classes of trees we can prove what the bases are, and we can use these classes as stop conditions in the algorithm.

Notice now that this method of enumeration defines a map $\mathbf{h}(\underline{p}) \longrightarrow \mathbf{h}(\mathcal{L}(\underline{p}))$ which takes a comb to the 'second level' comb in which all branches marked x_i are moved to 'very small legs' on the leg marked x_j $j \neq i$ most immediately to their right. The space $\mathbf{h}(\mathcal{L}(\underline{p}))$ is of course again isomorphic to $\mathbf{h}(\underline{p})$ by an earlier result, but let's pretend we don't know this. Then such maps define a filtration on the space of tree-like Jacobi diagrammes,

$$\mathbf{h}(\underline{p}) \longrightarrow \mathbf{h}(\mathcal{L}(\underline{p})) \longrightarrow \mathbf{h}(\mathcal{L}(\mathcal{L}(\underline{p}))) \longrightarrow \ldots$$

This allows us to take things one step further, and attach a homology theory to this filtration, in the spirit of Hochschild homology. As far as I know, this is a new structure on this space, and it generalizes naturally to higher loop degrees.

First we write an element of $\mathbf{h}(\mathcal{L}(\underline{p}))_{n+1}$ as $(a_0, a_1, \ldots, a_n)$, and then we define the "Tree-like Jacobi-Diagrammatic Hochschild Boundary" (the *tree boundary* for short) as

$$b := \sum_{i=0}^{n} (-1)^i d_i$$

where the operators $d_i : \mathcal{L}(\underline{p})^{\otimes n+1} \longrightarrow \mathcal{L}(\mathcal{L}(\underline{p}))^{\otimes n}$ are given by

$$(2.2)\qquad d_i(a_0, a_1, \dots, a_n) := \begin{cases} (a_0, \dots, [a_i, a_{i+1}], \dots, a_n) & \text{for } 0 \le i < n; \\ ([a_n, a_0], a_1, \dots, a_{n-1}) & \text{for } i = n. \end{cases}$$

Now we define the forgetful functor $G : \mathbf{h}(\mathcal{L}(\underline{p})) \longrightarrow \mathbf{h}(\overline{|\mathcal{L}(\underline{p})|})$ by forgetting the structure of $\mathcal{L}$. Combining this functor with the tree-boundary, we get a well-defined chain-complex whose homology groups we can calculate.

We give an interpretation of this boundary on the level of trees. To do this we must define a multiplicative structure on legs labelled by elements of A. Let a and b in A be mapped to legs labelled a and b. Then we map $a \star b \in F(A)$ the formal product of a and b to the trivelant vertex oriented from a to b and then to the stalk of the univelant vertex connected to a plus that connected to b. Given this definition, $\star$ can most suggestively by though of as the Lie bracket operation. In this way, the tree homology is a natural extension of the ideas listed above.

3. Loop Degree 0

As we know from the previous section, $\mathcal{A}'_c(p)_{-1}$ is closely related to free Lie algebras. Free Lie algebras are closely related to the algebra of circular words, and hence we should not find it surprising that the algebra of loop degree 0 Jacobi diagrammes $\mathcal{A}'_c(p)_0 \setminus \mathcal{A}'_c(p)_{-1}$ should turn out to be isomorphic as a $\Bbbk$ algebra to the $\Bbbk$ algebra of the quotient of the free group over A by the dihedral action.

Definition 3.1. Let $F_{\Bbbk}(A)$ be the free $\Bbbk$-algebra on a set A. We say that $u, v \in F_{\Bbbk}(A)$ are *conjugate* if there exist words $x, y \in F(A)$ such that $u = xy$ and $v = yx$. The *dihedral algebra over* A $\bigcirc(A)$ is the set of equivalence classes of $F_{\Bbbk}(A)$ under conjugation and reflection. An element of $F_{\Bbbk}(A)$ in the dihedral algebra is called a *dihedral necklace*.

Thus, by Pólya's theory of counting ([8]), the number of dihedral necklaces having n_i occurrences of the letter a_i for $1 \le i \le |A|$ ($n := \Sigma_{i=1}^3 n_i$) is the coefficient of $\prod_{j=1}^{i-1} X_j^{n_{j+1}}$ in $Z_{D_n}(1 + \sum_{j=1}^{i-1} X_j, 1 + \sum_{j=1}^{i-1} X_j^2, \dots, 1 + \sum_{j=1}^{n-1} X_j^n)$. Writing $T_m := 1 + \sum_{j=1}^{i-1} X_j^m$, this is given by the expression

$$Z_{D_n} = \begin{cases} \frac{1}{2n}\left(\sum_{d|n} \varphi(d) T_d^{n/d} + nT_1 T_2^{\frac{n-1}{2}}\right) & \text{for } n \text{ odd;} \\ \frac{1}{2n}\left(\sum_{d|n} \varphi(d) T_d^{n/d} + \frac{n}{2} T_1^2 T_2^{\frac{n}{2}-1} + \frac{n}{2} T_2^{\frac{n}{2}}\right) & \text{for } n \text{ even.} \end{cases}$$

Notice what we are doing here. Let's formalize it.

First, we propose a naming of Jacobi diagramme components, which the author hopes may become standard.

Definition 3.2. A choice of a maximum set of disjoint cycles in a Jacobi diagramme shall be called a *rim* of the diagramme. The set of all rims of a given diagramme D shall be denoted $rim(D)$. A connected subgraph in D with one vertex but no edges in r for $r \in rim(D)$ which is connected to the skeleton shall be called a *spike* of the diagramme, while a connected subgraph in D with no edge in common with a rim which does not connect to the skeleton shall be called a *spoke* of the diagramme.

These sets shall be denoted $spike_r(D)$ and $spoke_r(D)$ correspondingly. When there is no risk of confusion, the subscript r may be dropped.

Let us now adopt Chmutov and Duzhin's electro-technical interpretation, in which IHX is viewed as Kirchhoff's law ([2]). In this way of looking at things, for each Jacobi diagramme D and $r \in rim(D)$ we define $wire_r(D) := D - spike_r(D) + (r \cap spike_r(D))$ as the *wire for D*.For a fixed trivelant graph (perhaps with closed circle components) $wire_r(D)$, we view the spikes as electric particles flowing along the wire via IHX and AS relations which leave $wire_r(D)$ fixed. A Jacobi diagramme we get from D by the flow of the legs along $wire_r(D)$ we call *Jacobi Diagramme Flow State* of D.

Proposition 3.3. *The space of all Jacobi diagrammes with a fixed wire is isomorphic to the space of such diagrammes whose spikes are all legs.*

Remark 3.4. Note that in negative loop degree the claim is trivial.

This suggests that the correct tool to analyze Jacobi diagrammes given our approach is some sort of tool which fixes the wire and allows us to look only at the 'electric flow'. Here, perhaps the correct tool might be some sort of diagrammatic analogue to the homological algebraic structure of crossed simplicial homology ([7], Chapter 6), at least for low loop degrees where we can actually calculate such objects.

In the loop degree 0 case, the symmetry set of the circle is $SO(2)$ (in higher degrees it will be a finite group), and since the wire contains no trivelant vertices, which means that the 'electric flow' along the wire is just rotation and reflection, we get the free group modulo the $SO(2)$ action. Using the tree boundary above as if it were a Hochschild boundary in dihedral homology, we obtain the 'dihedral homology' of a free group.

4. Loop Degree 1

In loop degree 1, the wire is a Θ shape, which leads us to the denote the space of loop degree 1 Jacobi diagrammes over A a set with p elements as $\Theta(A)$. Let us arbitrarily label the three arcs in the wire 1, 2, and 3, and let us arbitrarily label the vertices a and b. This allows us to define a map from $F_{\Bbbk}(A)^{\otimes 3}$ to $\Theta(A)$ which sends $(w_1, w_2, w_3) \in F_{\Bbbk}(A)^{\otimes 3}$ to the loop degree 1 Jacobi diagramme where w_i gives the leaves of the arc labelled i when we read along the arc from a to b. Writing $w_i := \prod_{j=1}^{|w_i|} a_j^i$, the 'Jacobi diagramme flow' gives us the relation

$$\left(\prod_{j=1}^{|w_1|} a_j^1, \prod_{j=1}^{|w_2|} a_j^2, \prod_{j=1}^{|w_3|} a_j^3\right) = \left(\prod_{j=1}^{|w_1|} a_j^1, \prod_{j=1}^{|w_2|-1} a_j^2, (\prod_{j=1}^{|w_3|} a_j^3) \times a_{|w_2|}^2\right) - \left((\prod_{j=1}^{|w_1|} a_j^1) \times a_{|w_2|}^2, \prod_{j=1}^{|w_2-1|} a_j^2, \prod_{j=1}^{|w_3|} a_j^3\right) \tag{4.1}$$

These relations allow us to see that in fact a map from $F_{\Bbbk}(A)^{\otimes 2}$ to $\Theta(A)$ is sufficient. We shall call $F_{\Bbbk}(A)^{\otimes 2}$ modulo these two relations $\mathbf{H}(A)$(where the product is now the free product). Clearly $\mathbf{H}(A)$ is isomorphic to a generating set of $\Theta(A)$, which corresponds to loop degree 1 Jacobi diagrammes D where there are no legs connected to $spoke_r(D)$ (more precisely- where $wire_r(D) - rim_r(D)$ is the empty graph). The shape of the wire induces an D_2 symmetry on the corresponding $\mathcal{A}$-space, which of course corresponds to an D_2 symmetry on $\mathbf{H}(A)$.

Dihedral Homology is a well-known homological-algebraic structure (again Loday [7]), although we must generalize our definition of the diagrammatic 'Hochschild boundary' to a wire with a non-empty set of spokes.

Definition 4.1. Let D be a Jacobi diagramme with given rim r such that all legs of D have a vertex on $rim_r(D)$. Let us choose a starting point along the rim and an orientation of the rim, and label the legs $(a_0, a_1, \ldots, a_n)$ relative to that rim. Then the Jacobi Diagramme Hochschild boundary map on $wire_r(D)$ is defined as

$$b := \sum_L b_L$$

where the sum ranges over all loops L in the graph. For such a loop, containing n_L legs whose order is induced from the order of $(a_0, a_1, \ldots, a_n)$, b_L is defined to be

$$b_L := \sum_{i=0}^{n_L} (-1)^i d_i$$

where the operators $d_i : F_{\Bbbk}(\underline{p})^{\otimes n+1} \longrightarrow F_{\Bbbk}(\mathcal{L}(\underline{p}))^{\otimes n}$ are given by

$$d_i(a_0, a_1, \ldots, a_{n_L}) := (d \begin{cases} (a_0, \ldots, [a_i, a_{i+1}], \ldots, a_{n_L}) & \text{for } 0 \leq i < n_L; \\ ([a_{n_L}, a_0], a_1, \ldots, a_{n_L - 1}) & \text{for } i = n_L. \end{cases} \tag{4.2}$$

and d_i acts as the identity on all legs not in the loop L.

This gives us a Jacobi diagram Hochschild homology for all loop degrees, and again, we would like to modify it if possible to take into account wire symmetries. In this case, since the symmetry group is D_2, this corresponds to dihedral homology.

Although again, there is a direct approach to problems such as enumeration. The 'smallest' base we can hope for is that the legs on both sides be smaller than any dihedral group action on them, and that the smaller of the two words be on the arc labelled 1. We can do all reflections 'automatically', and associate with each element of the resulting generating set the vector $(m_1, m_3)_D \in \mathbb{N}^2$ where m_i is the minimum power of t the generator of the cyclic group, in absolute value, plus 1 if the word needs reflection, that would make $t^{m_i} w_i$ optimal in the sense above ($i = 1, 3$). The basis will thus be made up of diagrammes corresponding to those elements of $F_{\Bbbk}(A)^{\otimes 2}$ which have minimal $(m_1, m_3)_D$ in their equivalence class in $\mathbf{H}(A)$ and which are reflection-optimal in the sense given above, when order is determined first by sum and then alphabetically.

The key theoretical proof in the algorithm is to show that if we can raise complexity and then reduce it to below what it was originally, then we can replace this by reducing complexity all the time. Here, we use Whitehead's idea of *peak reduction*, which is still work in progress. The problem then becomes to check when two elements of equal complexity are taken from one to the other via an automorphism.

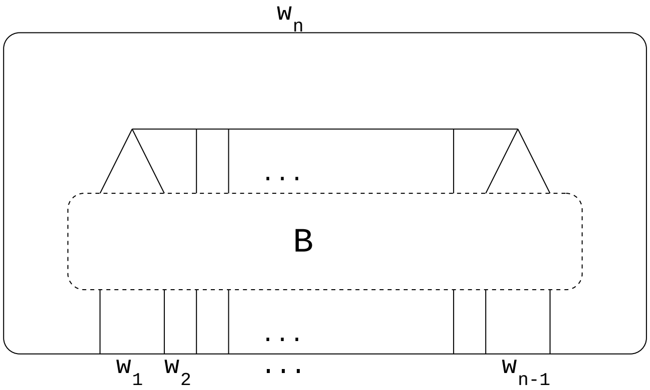

FIGURE 9. The general case, B a braid

Also, we can again use the idea we used in the tree case of passing from spokes that are legs to spokes that are combs with a marked head. Here, the key idea is the following exact sequence:

$$\Theta(\widetilde{\mathcal{L}(\underline{p})})^{n-1} \longrightarrow \Theta(\underline{p})^n \longrightarrow \Theta(\underline{p})^n/\Sigma_n \longrightarrow 0 \tag{4.3}$$

where the symmetric group acts on the legs, and $\Theta(\widetilde{\mathcal{L}(\underline{p})})^{n-1}$ is $\Theta(\mathcal{L}(\underline{p}))^{n-1}$ where in each word all free Lie algebra elements are of length 1 except for one which is of length 2. This space we understand via recursion as before, and the last space in the SES gives us the ability to order the legs as we like on each arc, so we know everything about that space as well. This allows us to recursively calculate the dimension in this loop degree. Note that the final element in our recursion is theta-shaped wires with a single spoke- and the space of these is isomorphic to the free Lie algebra on p generators.

5. The General Case

Again, we first 'sweep' all legs onto the rim, which we may assume to have a single connected component. $spoke_r(D)$ may now by IHX'd and AS'd until it is a comb, although we must watch out as we can say nothing about the cyclic order of its 'legs' along the rim. We can show that we can do this in such a way that $spoke_r(D)$ is a connected graph. We may have to change rims a few times along the way, and it is non-trivial that we may in fact bring an arbitrary Jacobi diagramme of positive loop degree to the form of figure 9.

In this way, we are looking at the wire as an element of $\mathcal{A}'_c(p)^0$, a Jacobi diagramme with no legs. Such spaces are researched in [1],[5]. It may be a reasonable conjecture that we may obtain a basis for our space by first taking a basis for such legless diagrammes and then flowing legs through them. This is known for loop degrees smaller than 4. Let us assume that this conjecture is true in general, and proceed. Note that the conjecture seems to imply that for $l > 0$, n, and p, $|\mathcal{A}'_c(\underline{p})^n_l|$ is divisible by $|\mathcal{A}'_c(\underline{p})^l|$.

Then we can choose one wire and see that in the general case, modulo reflection, the $\mathcal{A}$-space is given by 'optimizing' the words $(w_1, \ldots, w_l)$ in the sense of bringing them as close as possible to the minimum in the class of the dihedral group action on each word, by flowing the legs through the spoke which is just a comb.

Or using the second approach we used in the loop-degree 1 case, we have an exact sequence like Equation 4.3, where our problem is reduced to the calculation of the original space modulo the symmetric group action on the legs on each arc of the wire. This space seems much simpler than the original space. In fact, we can just take basis elements of $\mathcal{A}(S^1)$ and colour them with one of p colours, modulo arrangement on each arc and symmetries, and this may give us a basis- although this is a very optimistic conjecture.

In this way, the approach outlined above might perhaps give us some insight into the theory of $\mathcal{A}$-spaces in all degrees.

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