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ON THE CROSSING NUMBER OF KNOT AND THE CANONICAL GENUS OF ITS WHITEHEAD DOUBLE

TAKUJI NAKAMURA

Abstract. We show that the crossing number of a 2-bridge knot coincides with the canonical genus of its Whitehead double.

1. Introduction

A *link* is a smooth 1-manifold embedded in the 3-sphere S^3, and a *knot* is a link with one connected component. A *Seifert surface* of a knot K is a compact, connected, orientable surface S in S^3 such that the boundary of S is K. The minimal genus among all Seifert surfaces of K is called the *genus* for K, denoted by $g(K)$. A Seifert surface of K with the minimal genus is called a *minimal genus Seifert surface* of K. A Seifert surface of K is said to be *canonical* if it is obtained from a diagram of K by applying Seifert's algorithm. Then the minimal genus among all canonical Seifert surfaces of K is called the *canonical genus* for K, denoted by $g_c(K)$. A Seifert surface S of K is said to be *free* if the fundamental group of the complement of S, namely, $\pi_1(S^3 - S)$ is a free group. Then the minimal genus among all free Seifert surfaces of K is called the *free genus* for K, denoted by $g_f(K)$. For these "genus" of knots we have the following fundamental inequality, since any canonical Seifert surface is free,

$$g(K) \leq g_f(K) \leq g_c(K).$$

There are a lot of works constructing knots which gives the above inequality strictly. For the free genus and the genus, in 1972, H.C. Lyon [5] constructed a family of knots without free incompressible Seifert surfaces, hence $g(K) < g_f(K)$. In 1987, Y. Moriah [7] showed that there exists a knot K such that $g_f(K) - g(K) \geq n$ for any positive integer n. Subsequently, a similar result was showed by C. Livingston [6]. On the other hand, for the canonical genus, H.R. Morton showed that a twisted Whitehead double of the trefoil knot has the canonical genus at least three although its genus is one. Later, A. Kawauchi [2] showed that there exists a knot K such that $g_c(K) - g(K) = 2n$ for any positive integer n. After that, M. Kobayashi and T. Kobayashi [4] showed that there exists a knot K such that $g_c(K) - g_f(K) = n$ and $g_f(K) - g(K) = n$ for any positive integer n. Knots in these results are satellite knots or composite knots. The author [10] showed that there exists a simple fibered knot K such that $g_c(K) \geq n$ and $g_f(K) = 3$ for any positive integer $n(\geq 3)$. Shortly

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after, J.J. Tripp [14] showed that the canonical genus of a twisted Whitehead double of a torus knot of type $(2, n)$ is equal to n. Then he has conjectured that *the crossing number of a knot coincides with the canonical genus of its Whitehead double.*

We give a partial affirmative answer to this conjecture. In fact, we prove:

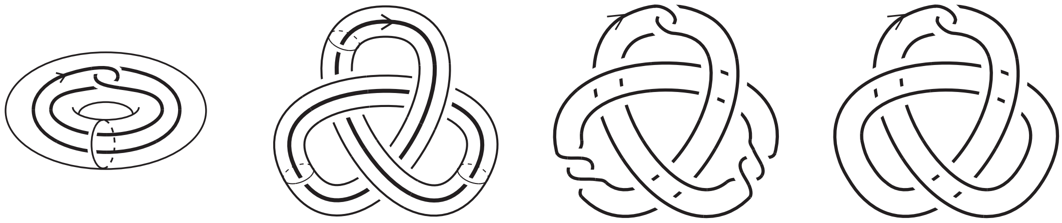
Theorem 1. *The crossing number of a* 2*-bridge knot coincides with the canonical genus of its Whitehead double.*

This note is organized as follows. In Section 2, we will prepare several definitions and notation, *Whitehead doubles*, *doubled links* and *Morton's inequality* ([8, Theorem 2]). In Sections 3 and 4, we will show that the canonical genus of a Whitehead double of a 2-bridge knot is equal to the crossing number of the 2-bridge knot by using Rudolph's technique in [13, Section 2].

Throughout this note, all manifolds in S^3 are oriented unless otherwise stated. For the definition of standard terms in knot theory, we refer to [1], [3], [9] and [12].

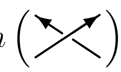
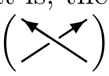
2. Preliminaries

2.1. **Doubles of knots and links.** Let C be a knot in an unknotted solid torus $S^1 \times B^2$ as in Figure 1 (a), called the Whitehead clasp, and $N(K)$ a tubular neighborhood of a nontrivial knot K in S^3 as in Figure 1 (b). Let $f : S^1 \times B^2 \to N(K)$ be an orientation preserving homeomorphism taking $\{0\} \times B^2$ to the meridian disk of $N(K)$, and $S^1 \times \{0\}$ to K. We call the knot $f(C)$ the *m-twisted Whitehead double* of K, denote by $D_m(K)$, if the linking number of $f(\ell)$ and K is equal to m, where ℓ is the preferred longitude of $S^1 \times B^2$.



(a) (b) (c) (d)

Figure 1

Let $w(P)$ be the writhe of a diagram P of a knot K, that is, the sum of the signs of all crossings in P, defined as *sgn*  $= 1$ and *sgn*  $= -1$. Then we see that the $w(P)$-twisted Whitehead double of K has a "nice" diagram, which is the 2-parallel diagram for P with a clasp. See Figure 1 (d). We denote by $D(P)$ this nice diagram of the $w(P)$-twisted Whitehead double of K.

Let L be a link with μ components $K_1, K_2, \ldots, K_\mu$ in S^3, and V_i $(i = 1, 2, \ldots, \mu)$ an unknotted solid torus $S^1 \times B^2$ containing a 2-component parallel link L_i with the opposite orientation as in Figure 2. Let $f_i : V_i \to N(K_i)$ be an orientation preserving homeomorphism taking the meridian disk of V_i to the meridian disk of $N(K_i)$, and the core of V_i, namely, $S^1 \times \{0\}$, to K_i. We call the link $f_1(L_1) \cup \cdots \cup f_\mu(L_\mu)$ the $(m_1, \ldots, m_\mu)$-*twisted doubled link* of L and denote it by $D_{(m_1,\ldots,m_\mu)}(L)$, if the linking number of $f_i(\ell_i)$ and K_i is equal to m_i, where ℓ_i is the preferred longitude of V_i for each i.

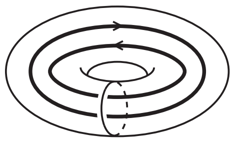


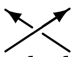
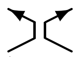
Figure 2

Let P be a diagram of L, and P_i the subdiagram of P corresponding to K_i for $i = 1, 2, \ldots, \mu$. Let w_i be the writhe of P_i. Similarly to the case of the Whitehead doubles of knots, we see that the $(w_1, \ldots, w_\mu)$-twisted doubled link of L has a "nice" diagram, which is the 2-parallel diagram for P. We denote by $D_L(P)$ this nice diagram of the $(w_1, \ldots, w_\mu)$-twisted doubled link of L.

2.2. **Morton's inequality and Canonical genus.** Let $P_L(v, z)$ be the HOMFLY polynomial of a link L calculated by the following recursive relations.

(1) $P_O(v, z) = 1$,

(2) $v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z)$,

where O is the trivial knot and L_+, L_- and L_0 are three links that are identical except near one point ,  and  respectively.

In [8], Morton showed the following inequality, called *Morton's inequality.* We denote the maximal degree in z of $P_L(v, z)$ by $\mathrm{maxdeg}_z P_L(v, z)$.

Theorem 2 ([8, Theorem 2])**.** *For a diagram D of a link L,*

$$\mathrm{maxdeg}_z P_L(v, z) \leq c(D) - s(D) + 1,$$

where $c(D)$ is the number of crossings and $s(D)$ is the number of Seifert circles in D, respectively.

The equality holds for alternating links, positive links and many other links. The right-hand side of Morton's inequality is the first Betti number of a canonical Seifert surface obtained from D. Thus the half of the maximal degree in z of $P_K(v, z)$ gives a lower bound for the canonical genus for a knot K, that is,

$\text{maxdeg}_z P_K(v,z) \leq 2g_c(K)$.

3. Lemmas

For the proof of Theorem 1, we must prove the following proposition.

Proposition 3. *Let $L_{m,n}$ be the doubled link of a 2-bridge link L, which has the diagram $D_L(P)$, where P is Conway's normal form $C(a_1, a_2, \ldots, a_m)$ of L such that $a_1 + a_2 + \cdots + a_m = n$ and $a_i > 0$ for $i = 1, 2, \ldots, m$. Then we have:*

$$\text{maxdeg}_z P_{L_{m,n}}(v,z) = 2n - 1.$$

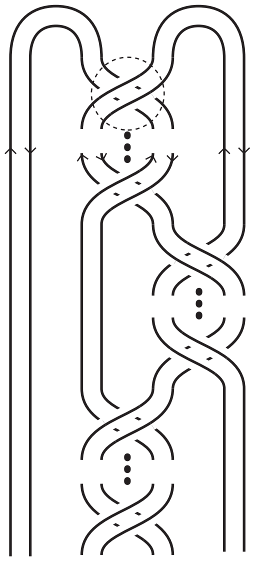


Figure 3

Note that $C(1, a_2, a_3, \ldots, a_m)$ is equivalent to $C(-a_2-1, -a_3, \ldots, -a_m)$. Hence we may assume, without loss of generality, that $a_1 \geq 2$. We note that the crossing number, the canonical genus and the maximal degree in z of $P_L(v,z)$ of a link L are the same as those of the mirror image of L. Similarly, we may assume that $a_m \geq 2$.

Let $k := n - m\,(> 1)$ be a positive integer. In order to prove Proposition 3 by induction on the lexicographic order of a pair (m,k) of positive integers m, k, we first prove the cases $(1,k)$, $(2,k)$ and $(m,2)$ respectively. The first case, $(1,k)$, has been proved by Tripp in [14, Proposition 1]. Thus we show the second case, $(m,2)$, that is, $a_1 = a_m = 2$ and $a_i = 1\,(2 \leq i \leq m-1)$ as follows. Hereafter, we denote by $d(L_{m,n})$ the maximal degree in z for $P_{L_{m,n}}(v,z)$ for short.

Lemma 4. *For the doubled link $L_{n-2,n}$ $(n \geq 3)$ of a 2-bridge link $C(2, 1, \ldots, 1, 2)$, we have:*

$$d(L_{n-2,n}) = 2n - 1.$$

Proof of Lemma 4. We prove Lemma 4 by induction on n. By direct calculations, we have $d(L_{1,3}) = 5$, $d(L_{2,4}) = 7$ and $d(L_{3,5}) = 9$.

Assume that Lemma 4 holds for every positive integer less than $n\,(\geq 6)$. We use a technique in [13, Section 2] to compute $P_{L_{n-2,n}}(v,z)$. By constructing a resolution tree with respect to the local diagram in the dotted circle depicted in Figure 3, we have eleven links $A_1^n, A_2^n, \ldots, A_7^n$ and $B_1^n, \ldots, B_4^n$ with diagrams identical to the diagram as in Figure 3 except as indicated in Figure 4. (The local diagram of $L_{n-2,n}$ in Figure 4 is added two crossings by Reidemeister move II.) We use this to compute $P_{L_{n-2,n}}(v,z)$ in a standard way. In the partial resolution tree as in Figure 4, the horizontal lines (resp. the vertical lines) are labeled vz or $-v^{-1}z$ (resp. v^2 or v^{-2}) according to the sign of the crossing which will be altered by a smoothing (resp. crossing change).

Then we have:

$$(*)\quad P_{L_{n-2,n}} = v^2z^2(P_{A_1^n} - P_{A_2^n} - P_{A_4^n} + P_{A_6^n}) - z^2(P_{A_3^n} + P_{A_5^n}) + P_{A_7^n} + v^{-1}z(P_{B_1^n} + P_{B_2^n}) - vz(P_{B_3^n} + P_{B_4^n}).$$

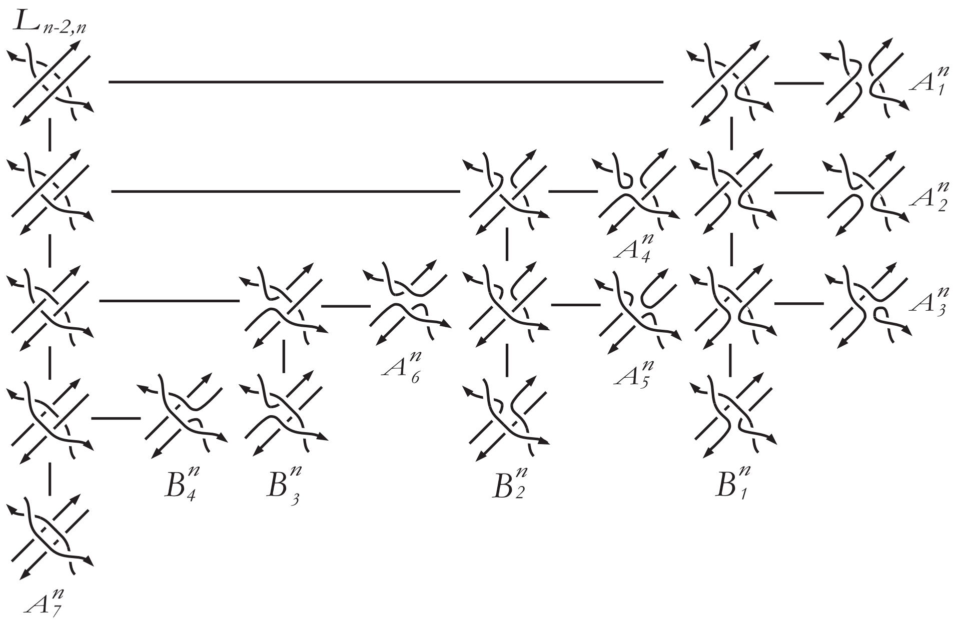


Figure 4

Claim 5. $d(A_i^n) \leq 1$ *for* $i = 2,3,4,5$.

Proof of Claim 5. We can deform each A_i^n into a diagram of a 2-component link which is the boundary of an unknotted, twisted annulus for $i = 2,3,4,5$. Since the canonical Seifert surface obtained from the diagram, namely, the annulus has the first Betti number one, the conclusion follows from Morton's inequality. □

Hence we see that none of $A_2^n, \ldots, A_5^n$ contributes anything to $d(L_{n-2,n})$ by induction assumption and the equality $(*)$ when $n \geq 6$. For $i = 1, 2, 3, 4$, we have then

$$d(L_{n-2,n}) \leq \max\{d(A_1^n) + 2, d(A_6^n) + 2, d(A_7^n), d(B_i^n) + 1\}. \tag{1}$$

By some deformations, it is easily seen that A_i^n $(i = 1, 6, 7)$ is the t_i-twisted doubled link of a 2-bridge knot K or the (t_i, t_i')-twisted doubled link of a 2-bridge link L for some integers t_i and t_i'.

Claim 6. *For any integers t_i and t_i', we have $d(A_1^n) = d(L_{n-3,n-1})$, $d(A_6^n) = d(L_{n-4,n-2})$ and $d(A_7^n) = d(L_{n-5,n-3})$.*

Proof of Claim 6. We prove Claim 6 only for the case where A_i^n is the t_i-twisted doubled link of a 2-bridge knot K_i. The other case can be proved similarly.

We see that K_i has a diagram D_i of Conway's normal form $C(2, a_1, a_2, \ldots, a_{m_i}, 2)$ or $C(-2, -a_1, -a_2, \ldots, -a_{m_i}, -2)$, where $a_j = 1$ for any j and $m_1 = n-5$, $m_6 = n-6$ and $m_7 = n - 7$. Let w_i be the writhe of D_i. If $t_i = w_i$, we have the conclusion obviously. Suppose $t_i - w_i > 0$. (The case $t_i - w_i < 0$ can be proved similary.) For A_i^n $(i = 1, 6, 7)$ by a skein relation for the crossing in the dotted circle in Figure 5, we have:

$$P_{A_i^n} = v^{-2} P_{L'} - v^{-1} z P_{L''}$$

for certain links L' and L''.

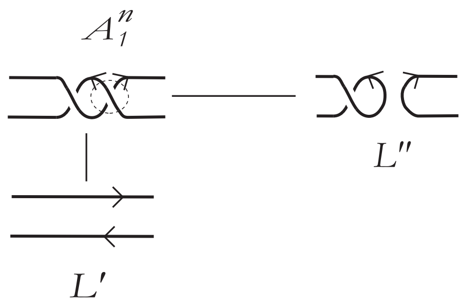
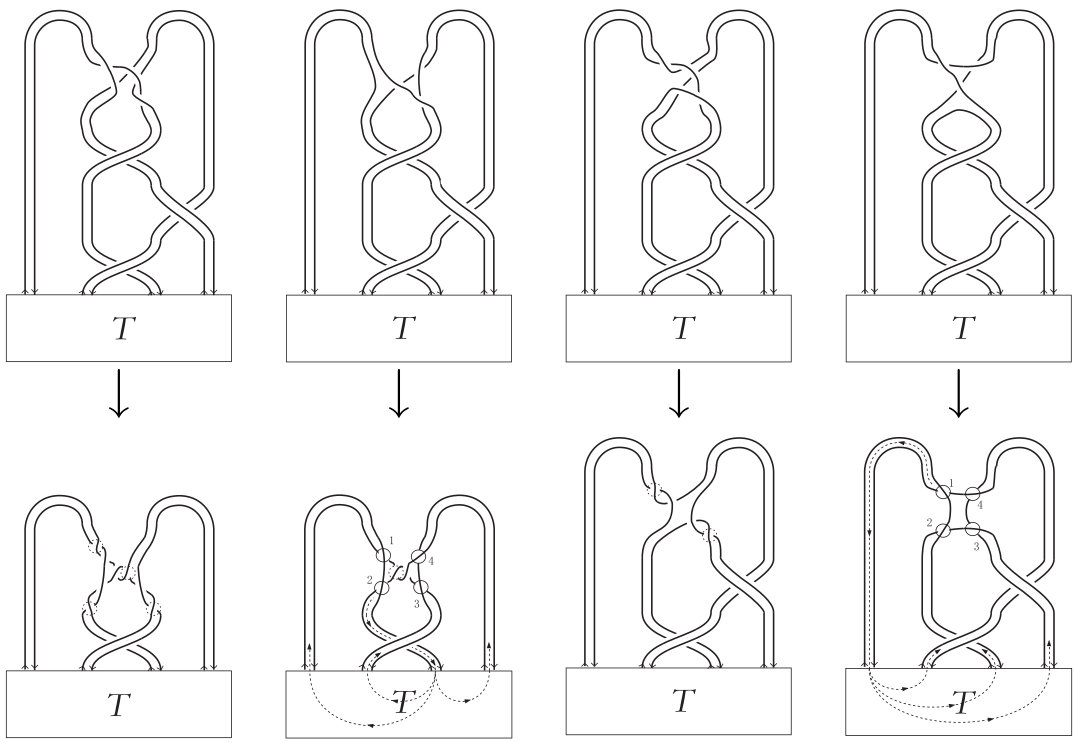


Figure 5

Then we see that L'' is equivalent to the trivial knot. (In the case where A_i^n is a (t_i, t_i')-twisted doubled link of a 2-bridge link, L'' is the 3-component trivial link or a 3-component link which is the split union of the trivial knot and the boundary of an unknotted, twisted annulus.) On the other hand, L' is $(t_i - 1)$-twisted doubled link of K and $d(L') = d(A_i^n)$. By repeating this procedure if necessary, we obtain the mirror image of $L_{n-3,n-1}$ from A_1^n, $L_{n-4,n-2}$ from A_6^n and the mirror image $L_{n-5,n-3}$ from A_7^n, respectively as the result of crossing changes. Hence we have the conclusion by induction assumption. □

Claim 7. $d(B_i^n) \leq 2n - 6$ *for* $i = 1, 2, 3, 4$.

Proof of Claim 7. First we deform the upper diagram B_i^n $(i = 1, 2, 3, 4)$ into the lower diagram as in Figure 6, where each rectangle contains the same tangle T.



(a) B_1^n (b) B_2^n (c) B_3^n (d) B_4^n

Figure 6

Now we consider the lower diagrams of B_1^n and B_3^n. Then by performing crossing changes at the crossings in the dotted circles in Figure 6 (a) (resp. Figure 6 (c)), we obtain a diagram with $2n-3$ Seifert circles and $4n-12$ crossings from the lower diagram of B_1^n (resp. a diagram with $2n-1$ Seifert circles and $4n-8$ crossings from the lower diagram of B_3^n). On the other hand, smoothing at each crossing yields a 2-component link which is the boundary of an unknotted, twisted annulus (or a link with $d(L_{n-5,n-3})$ for B_1^n).

For the lower diagram of B_2^n, we consider the crossings labeled 1,2,3 and 4 as in Figure 6 (b). Then by moving the crossing 2 along the dotted line as in Figure 6 (b), we see that the crossing 2 and one of the other labeled crossings are cancelled wherever the crossing 2 reaches. Hence we see that the other pair of crossings a, b say, forms a right-handed full-twist. Then by performing crossing changes at either a or b and the crossing in the dotted circle in Figure 6 (b), we obtain a diagram with $2n-3$ Seifert circles and $4n-12$ crossings from the lower diagram of B_2^n. On the other hand, smoothing at each crossing yields also a 2-components link which is the boundary of an unknotted, twisted annulus, or a link with $d(L_{n-5,n-3})$.

For the lower diagram of B_4^n, we also consider the crossings labeled 1,2,3 and 4 as in Figure 6 (d). Note that the crossings labeled 1 and 4 are positive, and the crossings labeled 2 and 3 are negative. Hence there are two cases to be considered.

Case 1. The crossing 1 and the crossing 2 or 3 are cancelled by moving the crossing 1 along the dotted line as in Figure 6 (d). Then we see that the other crossings are also cancelled.

Case 2. The crossing 1 and the crossing 4 form a right-handed full-twist by moving the crossing 1 along the dotted line as in Figure 6 (d). Then the other pair of crossings 2 and 3 forms a left-handed full-twist. In this case, we perform crossing changes at the crossings 1 and 2.

In both cases we obtain a diagram with $2n-1$ Seifert circles and $4n-8$ crossings. Smoothing at each crossing yields also a 2-components link which is the boundary of an unknotted, twisted annulus.

Then we obtain, by Morton's inequality and induction assumption,

$$\begin{aligned} d(B_i^n) &\leq \max\{(4n-12)-(2n-3)+1, (4n-8)-(2n-1)+1, d(L_{n-5,n-3})+1\} \\ &= \max\{2n-8, 2n-6, 2(n-3)-1+1\} \\ &= 2n-6. \end{aligned}$$

The proof of Claim 7 is completed. □

By inequality (1) and Claim 7, we have

$$d(L_{n-2,n}) \leq \max\{d(A_1^n)+2, d(A_6^n)+2, d(A_7^n), (2n-6)+1\}. \tag{2}$$

Since $d(A_1^n)$, $d(A_6^n)$ and $d(A_7^n)$ are equal to $d(L_{n-3,n-1})$, $d(L_{n-4,n-2})$, $d(L_{n-5,n-3})$, respectively, by Claim 6, it follows from induction assumption

$$\begin{aligned} d(L_{n-2,n}) &\leq \max\{d(L_{n-3,n-1})+2, d(L_{n-4,n-2})+2, d(L_{n-5,n-3}), 2n-5\}, \\ &= \max\{2(n-1)-1+2, 2(n-2)-1+2, 2(n-3)-1, 2n-5\}, \\ &= \max\{2n-1, 2n-3, 2n-7, 2n-5\}. \end{aligned}$$

Since there exist the terms in $P_{L_{n-2,n}}$ whose degree in z is $2n-1$, we obtain $d(L_{n-2,n}) = 2n-1$. This completes the proof of Lemma 4. □

We can prove the third case $(2, k)$, that is, the following lemma similarly.

Lemma 8. *For the doubled link $L_{2,n}$ of a 2-bridge link $C(a_1, a_2)$, we have:*

$$d(L_{2,n}) = 2n-1.$$

□

We can also prove Proposition 3. We omit their proofs here. See our preprint [11].

4. Proof of Theorem 1

In this section, we prove our main theorem, Theorem 1.

Proof of Theorem 1. Let K be a Whitehead double of a 2-bridge knot $C(a_1, \ldots, a_m)$ with $a_i > 0$ for any i and $a_1 + \cdots + a_m = n$. We see that the genus of a canonical Seifert surface obtained from the diagram of K as in Figure 8 is equal to n. (Although the diagram as in Figure 8 is a diagram of a Whitehead double of $C(3,1,1,2,2)$, we can easily see the general case. We note that this kind of diagram was appeared in [2] for the trefoil knot and also observed by Tripp for the other torus knots of $(2, n)$ in [14].)

At the crossing in the Whitehead clasp as indicated in Figure 7, we perform a crossing change and a smoothing. Then by a skein relation, we have

$$P_K(v,z) = v^2 P_O(v,z) + vzP_L,$$

where O is the trivial knot and L is a doubled link of $C(a_1,\ldots,a_m)$. Then we see that $d(L) = d(L_{m,n}) = 2n-1$ by applying the argument similar to that in the proof of Claim 6 to the twists in the dotted rectangle in Figure 8. Hence we obtain that the maximal degree in z of $P_K(v,z)$ is equal to $2n$ by Proposition 3. Therefore the canonical genus of K is equal to n by Morton's inequality. The proof is now completed. □

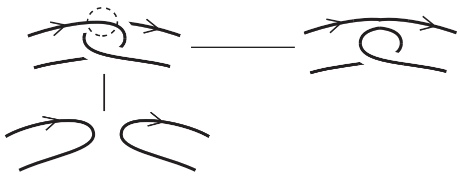


Figure 7

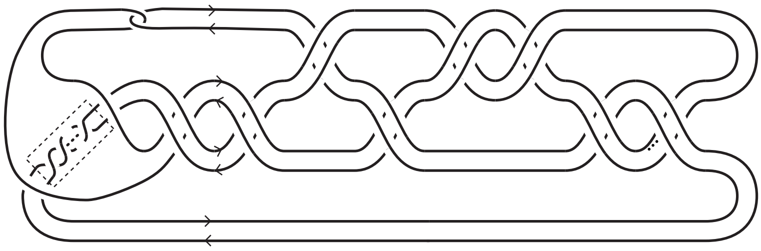


Figure 8

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Osaka City University Advanced Mathematical Institute, Sugimoto3-3-138, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail address: takuji@sci.osaka-cu.ac.jp