Proceedings of the East Asian School
of Knots, Links, and Related Topics
February 16–20, 2004 • Seoul, Korea

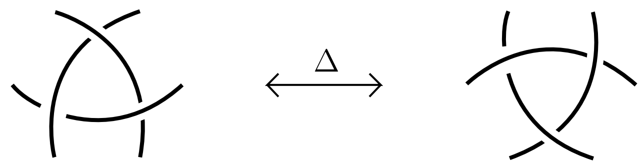
DELTA UNKNOTTING OPERATION AND VASSILIEV INVARIANTS

YASUTAKA NAKANISHI AND YOSHIYUKI OHYAMA

ABSTRACT. After their works of Goussarov and Habiro, it is known that a local move called C_n-move is strongly related to Vassiliev invariants of order less than n. Let K be a knot, and K^{C_n} the set of knots obtained from a knot K by a single C_n-moves. Let $\mathcal{V}_m$ be the set of Vassiliev invariants of order less than or equal to m ($m \geq 2$), and $\mathcal{V}_m\mathcal{K}$ the value set $\{(v, \{v(K)\}_{K\in\mathcal{K}})\}_{v\in\mathcal{V}_m}$ for a set of knots $\mathcal{K}$. Our main result is the following: If m_1, m_2 are sufficiently greater than n, then there exists a pair of knots K_1, K_2 such that $\mathcal{V}_{m_1}K_1 = \mathcal{V}_{m_1}K_2$, and $\mathcal{V}_{m_2}K_1^{C_n} \neq \mathcal{V}_{m_2}K_2^{C_n}$. In other words, the C_n Gordian complex is not homogeneous with respect to Vassiliev invariants. In this note, we will study the case $n = 2$. Here, a C_2-move is a Δ unknotting operation.

1. Introduction

For two link diagrams K and L which differ only in one place as in Fig. 1, a local move between K and L is called a Δ unknotting operation, or briefly a Δ move. It is known the following result by Matveev [Ma] and by Murakami and the first author [MN].



Proposition 1. *Two knots can be transformed into each other by a finite sequence of Δ moves.*

For a knot invariant v which takes value in some abelian group, v can be extended to an invariant of singular knots, where a singular knot is an immersion of a circle into $\mathbf{R}^3$ whose songularities are transversal double points. An invariant v is called a Vassiliev invariant of order n, if n is the smallest integer such that v vanishes on all singular knots with n double points or more. If a knot invariant is a Vassiliev invariant of order m for some integer m, it is called an invariant of finite type. The Vassiliev invariant of order 0 and 1 is known to be trivial. The coefficient of z^n term in the Conway polynomial is known to be a Vassiliev invariant of order n. After their works of Goussarov [G] and Habiro [H], it is known that a local move called C_n-move is strongly related to Vassiliev invariants of order less than n as follows. It is a generalization of Proposition 1.

Key words and phrases. Delta unknotting operation, Vassiliev invariant, Conway polynomial.

Proposition 2. *Two knots have the same value for each Vassiliev invariant of order less than n if and only if the two knots can be transformed into each other by a finite sequence of C_n-moves.*

The following fact is an observation on a topic whether such a relationship between C_n-moves and Vassiliev invariants is natural or not. We consider the case of a C_2-move (Δ move).

Theorem 3. (1) *For j polynomials with variables z, $\nabla_i(z) = 1 + a_2 z^2 + a_4^{(i)} z^4 + \cdots + a_{2n_j}^{(i)} z^{2n_j}$ $(1 \leq i \leq j)$, there exists a pair of knots K_1 and K_2 such that $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, $\nabla_{K_1^\Delta}(z) \not\ni \nabla_1(z), \ldots, \nabla_j(z)$, and $\nabla_{K_2^\Delta}(z) \ni \nabla_1(z), \ldots, \nabla_j(z)$.*
(2) *If integers m_1 and m_2 are sufficiently greater than n, there exists a pair of knots K_1 and K_2 such that $\mathcal{V}_{m_1} K_1 = \mathcal{V}_{m_1} K_2$, and that $\mathcal{V}_{m_2} K_1^\Delta \neq \mathcal{V}_{m_2} K_2^\Delta$.*

We remark that there are no dependence between integers m_1 and m_2. If we take m_1 greater, then the condition becomes stronger. If we take m_2 less (but necessarily greater than n), the condition becomes stronger. The reason why the coefficients of z^2 terms in $\nabla_i(z)$'s are the same is from the Okada observation in [Ok].

2. Surgical Description

It is well-known that any knot can be transformed to a trivial knot by crossing-changes at suitable crossing points. Every crossing-change is obtained by a ± 1 surgery along a small trivial knot around the crossing point with linking number 0. Levine [L] and Rolfsen [R1, R2] introduced a surgery description of a knot as follows:

Proposition 4. *Let K be a knot, K_0 a trivial knot. Then, there exist n disjoint solid tori $T_1, \ldots, T_n$ in $S^3 - K_0$ and a homeomorphism ϕ from $S^3 - {}^\circ T_1 \cup \cdots \cup {}^\circ T_n$ to itself such that*
(1) $\phi(K_0) = K$,
(2) $T_1 \cup \cdots \cup T_n$ *is a trivial link,*
(3) $\mathrm{lk}(T_i, K_0) = \mathrm{lk}(T_i, K) = 0$ *for each i,*
(4) $\phi(\partial T_i) = \partial T_i$ *and* $\mathrm{lk}(\mu_i', T_i) = 1$ *where* $\mu_i \subset \partial T_i$ *is a meridian of* T_i *and* $\mu_i' = \phi^{-1}(\mu_i)$.

From a surgery description, we have a surgical view of Alexander matrix of the knot as follows:

Proposition 5. *Let K be a knot. Then, K has an Alexander matrix $M_K = (m_{ij}(t))$ of the following form:*
(1) $m_{ij}(t) = m_{ji}(t^{-1})$,
(2) $|m_{ij}(1)| = \delta_{ij}$,
where the Kronecker's delta $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

3. Proof of Theorem 3

It is known that the relationship between the Alexander polynomial and the Conway polynomial for a knot K: $\nabla_K(t^{-1/2} - t^{1/2}) = \Delta_K(t)$. Let $\Delta_i(t) = \nabla_i(t^{-1/2} - t^{1/2})$

$(1 \leq i \leq j)$. It is also known that any Alexander polynomial can be realized by a knot with unknotting number 1 (see [K], [S]).

For the polynomial $\nabla_{j+1}(z) = 1 - ja_2z^2$, let $\Delta_{j+1}(t) = \nabla_{j+1}(t^{-1/2} - t^{1/2})$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = (\Delta_1\Delta_2\cdots\Delta_j\Delta_{j+1})^2$. For the polynomial $\nabla_{j+2}(z) = 1 - (a_2 \pm 1)z^2$, let $\Delta_{j+2}(t) = \nabla_{j+2}(t^{-1/2} - t^{1/2})$. Let K^{**} be a knot with unknotting number 1 and $\Delta_{K^{**}}(t) = \Delta_{j+2}(t)$. Let $K_1 = K^*\#K^*\#K^*\#K^*\#K^{**}$. Then, K_1 has a surgical view of Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 & 0 & 0 \\ 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 & 0 \\ 0 & 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 \\ 0 & 0 & 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 \\ 0 & 0 & 0 & 0 & \Delta_{j+2}(z) \end{pmatrix}.$$

A Δ move is realized by twice crossing changes. If K_1' is obtained from K_1 by a single Δ move, then K_1' is obtained from K_1 by twice crossing changes. Therefore, K_1' has a surgical view of Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 & 0 & 0 & * & * \\ 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 & 0 & * & * \\ 0 & 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & 0 & * & * \\ 0 & 0 & 0 & \prod_{i=1}^{j+1}\Delta_i(z)^2 & 0 & * & * \\ 0 & 0 & 0 & 0 & \Delta_{j+2}(z) & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}.$$

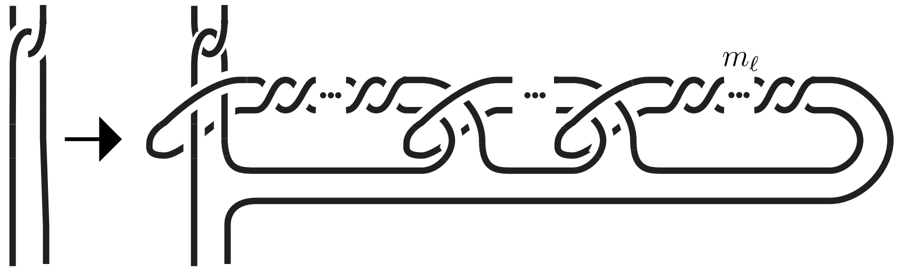
If $\Delta_{K_1'}(t) = \Delta_i(t)$, then we have the determinant of the above matrix is $\pm\Delta_i(t)$. We consider the equation modulo $(\Delta_i(t))^2$, which becomes a contradiction. Therefore, we have $\nabla_{K_1^\Delta}(z) \not\ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$.

Let K_2 be a knot with unknotting number 1 and $\Delta_{K_2}(t) = \Delta_{K_1}(t)$, (or a knot with unknotting number 1 and $\mathcal{V}_{m_1}K_1 = \mathcal{V}_{m_1}K_2$ (see [OTY], [Oh])). By the following Lemma, it can be seen that $\nabla_{K_2^\Delta}(z) \ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$. Hence the proof is complete.

Lemma 6. *Let K be a knot with algebraic unknotting number* 1*. For a set of integers $a'_2 = a_2(K) \pm 1$, and arbitrary integers a'_{2i} $(i = 2, 3, \ldots, \ell)$, there exists a knot $K' \in K^{\Delta}$ with $\nabla_{K'}(z) = 1 + a'_2 z^2 + a'_4 z^4 + \cdots + a'_{2\ell} z^{2\ell}$.*

Here, a knot with algebraic unknotting number 1 means that a single crossing-change yields a knot with the trivial Alexander polynomial (cf. [Mk1]).

Proof of Lemma. Since K is a knot with algebraic unknotting number 1, there exists a crossing at which the crossing-change yields a knot with the trivial Alexander polynomial. We consider such a crossing as in the left of Fig. 2. We transform this part of K to the right of Fig. 2. Here, $m_2, \ldots, m_\ell$ are numbers of left-handed full-twists. In a negative case, it means $|m_i|$ right-handed full-twists. By the parallel argument to that in [Mk2], the difference of the Conway polynomials is $z^2 - (m_2 + 1)z^4 \cdots + (-1)^{\ell-2}(m_{\ell-1} + 1)z^{2\ell-2} + (-1)^{\ell-1} m_\ell z^{2\ell}$. The proof is complete.



References

[G] Goussarov, *On n-equivalence of knots and invariants of finite degree*, Topology of Manifolds and Varieties, Ed. O. Viro, Amer. Math. Soc., Province (1994), 173–192.

[H] K. Habiro *Claspers and finite type invariants of links*, Geom. Topol. **4** (2000), 1–83.

[HY] M. Hirasawa and Y. Uchida, *The Gordian complex of knots*, J. Knot Theory Ramif. **11** (2002), 363–368.

[K] H. Kondo, *Knots of unknotting number* 1 *and their Alexander polynomials*, Osaka J. Math. **16** (1979), 551–559.

[L] J. Levine, *A characterization of knot polynomials*, Topology **4** (1965), 135–141.

[Ma] S. V. Matveev, *Generalized surgeries of three-dimensional manifolds and representations of homology sphere*, Russian, Mat. Zametki **42** (1987), 268–278, 345 English translation: Math. Notes **42** (1987), 651–656.

[Mk1] H. Murakami, *Algebraic unknotting operation*, Q & A in General Topology **8** (1990), 283–292.

[Mk2] H. Murakami, *Delta-unknotting number and the Conway polynomials*, Kobe J. Math. **10** (1993), 17–22.

[MN] H. Murakami, Y. Nakanishi *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89.

[Oh] Y. Ohyama, *Web diagrams and realization of Vassiliev invariants by knnots*, J. Knot Theory Ramif. **9** (2000), 693–701.

[Ok] M. Okada, *Delta-unknotting operations and the second coefficients of the Conway polynomial*, J. Math. Soc. Japan **42** (1990), 713–717.

[OTY] Y. Ohyama, K. Taniyama, and S. Yamada, *Realization of Vassiliev invariants by unknotting number one knots*, Tokyo J. Math. **25** (2002), 17–31.

[R1] D. Rolfsen, *A surgical view of Alexander's polynomial*, Geometric Topology (Proc. Park City, 1974), Lecture Notes in Math. **438** Springer-Verlag, Berlin and New York (1974), 415–423.

[R2] D. Rolfsen, Knots and Links, Math. Lecture Series **7** Publish or Perish Inc., Berkeley (1976).
[S] T. Sakai, *A remark on the Alexander polynomials of knots*, Math. Seminar Notes Kobe Univ. **5** (1977), 451–456.

Department of Mathematics, Faculty of Science, Kobe University, Rokko, Nada-ku, Kobe 657-8501, Japan
E-mail address: nakanisi@math.kobe-u.ac.jp

Department of Mathematics, College of Arts and Sciences, Tokyo Woman's Christian University, Zempukuji, Suginami-ku, Tokyo 167-8585, Japan
E-mail address: ohyama@lab.twcu.ac.jp