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# VIRTUAL KNOTS AND VIRTUAL CROSSINGS

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## 1. Introduction

For an oriented virtual link, Kauffman defined the $f$-polynomial (Jones polynomial). The supporting genus of a virtual link diagram is the minimal genus of a surface in which the diagram can be embedded. In this paper we show that the span of the $f$-polynomial of an alternating virtual link $L$ is determined by the number of crossings of any alternating diagram of $L$ and the supporting genus of the diagram. It is a generalization of Kauffman and Murasugi's theorem. Furthermore we prove that any virtual link diagram that is obtained from an alternating virtual link diagram by virtualizing one real crossing is not equivalent to a classical link diagram.

This is a short summary of [7]. Refer to [7] for the proofs and details. I would like to thank the organizers for giving an opportunity to attend the conference.

## 2. Results

An (oriented) *virtual link diagram* is a closed (oriented) 1-manifold generically immersed in $\mathbf{R}^2$ such that each double point is labeled to be either (1) a *real* crossing which is indicated as usual in classical knot theory or (2) a *virtual* crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Figure 1 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. A *virtual link* is the equivalence class of a virtual link diagram, cf. [1, 12, 13]. Unless otherwise stated, we assume that a virtual link is oriented.

Kauffman defined the $f$-polynomial $f_L(A) \in \mathbf{Z}[A, A^{-1}]$ of a virtual link. It is also called the normalized bracket polynomial or the Jones polynomial, cf. [13]. The *span* of $f_L(A)$ is the maximal degree of $f_L(A)$ minus the minimal. It is an invariant of a virtual link. We denote it by span$(L)$ or span$(D)$.

By $c(D)$, we mean the number of real crossings of $D$.

**Theorem 1** (Kauffman [10], Murasugi [15]). *Let $L$ be an alternating link represented by a proper alternating connected link diagram $D$. Then we have*

$$\text{span}(L) = 4c(D).$$

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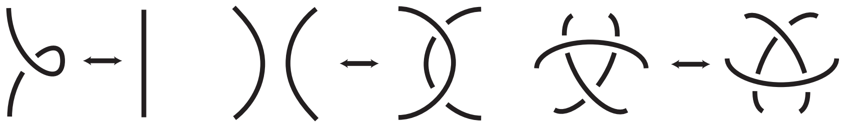

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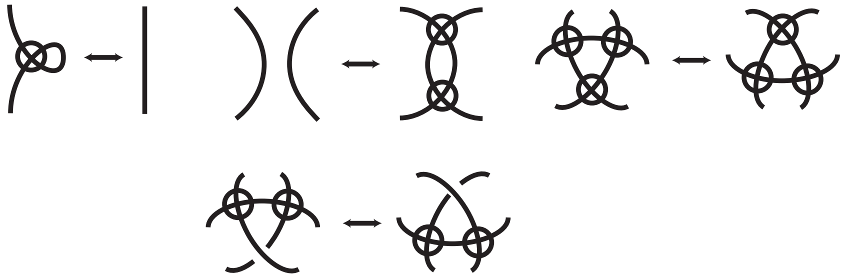
Reidemeistermoves



virtualReidemeistermoves

Figure 1

Any virtual link diagram $D$ can be realized as a link diagram in a closed oriented surface (cf. [13]). The *supporting genus* $g(D)$ of $D$ is the minimal genus of a closed oriented surface in which the diagram can be realized.

Note that $g(D)$ can be calculated. Consider a link diagram $\mathcal{D}$ in a closed oriented surface $F$ that realizes $D$. If some regions of the complement of $\mathcal{D}$ in $F$ are not open disks, replace them with open disks. Then we obtain a link diagram realizing $D$ in a surface of genus $g(D)$. We may also use a formula in Lemma 8.

Let $D$ be a virtual link diagram. By forgetting crossing information, it is the union of immersed circles, say $C_1, \cdots, C_\mu$ (for some $\mu \in \mathbf{N}$). Consider an equivalence relation that is the transitive closeure of binary $C_i \sim C_j$ where $C_i \sim C_j$ meant that $C_i \sim C_j$ has at least one real crossing. Then, for an equivalence class $\{C'_1, \cdots, C'_\lambda\}$, the restriction of $D$ to $C'_1 \cup \cdots \cup C'_\lambda$ is called a *connected component* of $D$. When $D$ is a connected component of itself, we say that $D$ is *connected*.

Since the supporting genus of a classical link diagram is zero, the following theorem is a generalization of Theorem 1.

**Theorem 2** (cf. [2], [3], [4], [5])**.** *Let $L$ be an alternating virtual link represented by a proper alternating connected virtual link diagram $D$. Then we have*

$$\mathrm{span}(L) = 4(c(D) - g(D)).$$

This theorem was announced in [2] and proved in [3] for an abstract link diagram instead of a virtual link diagram. It is proved in [8] that there exists a canonical correspondence between abstract links and virtual links (recalled in Section 3). In [4] and [5], this theorem was announced and proved for a link diagram in a surface. The ideas of the proofs in [3] and [5] are essentially the same, which can be applied to a virtual link. Furthermore we have the following.

**Theorem 3.** *Let $L$ be an alternating virtual link represented by a proper alternating virtual diagram $D$. Then we have*

$$\operatorname{span}(L) = 4(c(D) - g(D) + m - 1),$$

*where $m$ is the number of the connected components of $D$.*

When a virtual link diagram $D'$ is obtained from another diagram $D$ by replacing a real crossing $p$ of $D$ with a virtual crossing, then we say that $D'$ is obtained from $D$ by *virtualizing* the crossing $p$.

A virtual link diagram $D$ is said to be a *v-alternating* if $D$ is obtained from a proper alternating virtual link diagram by virtualizing one real crossing.

**Theorem 4** (Kishino[14])**.** *Let $D$ be a connected v-alternating virtual link diagram which is obtained from a proper alternating classical link diagram by virtualizing one crossing. Then*

$$span(D) = 4c(D) - 2.$$

We strengthen Kishino's theorem as follows:

**Theorem 5.** *Let $D$ be a v-alternating virtual link diagram. Then we have*

$$\operatorname{span}(D) = 4(c(D) - g(D) + m - 1) + 2,$$

*where $m$ is the number of connected components of $D$.*

*In particular, if $D$ is a connected v-alternating virtual link diagram, then*

$$\operatorname{span}(D) = 4(c(D) - g(D)) + 2.$$

When $D$ is a connected v-alternating virtual link diagram which is obtained from a proper alternating classical link diagram by virtualizing a crossing, the supporting genus $g(D)$ is 1. Thus Theorem 4 is a special case of Theorem 5.

**Corollary 6.** *Let $D$ be a v-alternating virtual link diagram. Then $D$ is not equivalent to a classical link diagram.*

## 3. Definitions

Let $D$ be an unoriented virtual link diagram. The replacement of the diagram in a neighborhood of a real crossing as Figure 2 are called A-splice and B-splice, respectively (cf. [10], [11]).

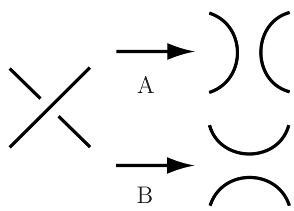



Figure 2

A *state* of $D$ is a virtual link diagram obtained from $D$ by doing A-splice or B-splice at each real crossing of $D$. The *Kauffman bracket polynomial* $< D >$ of $D$ is defined by

$$< D >= \sum_S A^{\natural(S)}(-A^2 - A^{-2})^{\sharp(S)-1},$$

where $S$ runs over all states of $D$, $\natural(S)$ is the number of A-splice minus that of B-splice used to obtain the state $S$, and $\sharp(S)$ is the number of loops of $S$.

For an oriented virtual link diagram $D$, the *writhe* $\omega(D)$ is the number of positive crossings minus that of negative crossings of $D$. The *f-polynomial* of $D$ is defined by

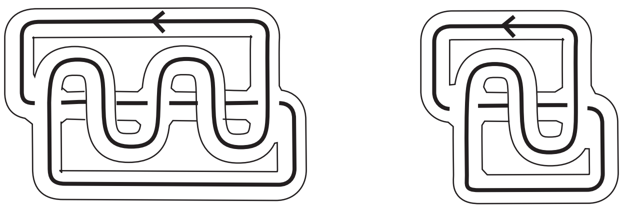
$$f_D(A) = (-A^3)^{-\omega(D)} < D > .$$

**Theorem 7** ([13]). *The f-polynomial is an invariant of a virtual link.*

For a virtual link $L$ represented by $D$, the $f$-polynomial $f_L(A)$ of $L$ is defined by $f_D(A)$. When $L$ is a classical link, the $f$-polynomial $f_L(A)$ is equal to the Jones polynomial $V_L(t)$ after substituting $A^4$ for $t$.

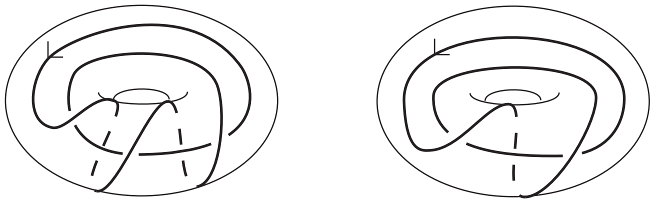
A pair $P = (\Sigma, \mathcal{D})$ of a compact oriented surface $\Sigma$ and a link diagram $\mathcal{D}$ in $\Sigma$ is called an *abstract link diagram* (ALD) if $|\mathcal{D}|$ is a deformation retract of $\Sigma$, where $|\mathcal{D}|$ is a graph obtained from $\mathcal{D}$ by replacing each crossing with a vertex. If $\mathcal{D}$ is an oriented link diagram, then $P$ is said to be *oriented*. Unless otherwise stated, we assume that an ALD is oriented. If $|\mathcal{D}|$ is connected (or equivalently, $\Sigma$ is connected), then $P$ is said to be *connected*. Two examples of connected ALDs are illustrated in Figure 3 (a) and (b).

Let $P = (\Sigma, \mathcal{D})$ be an ALD. For a closed oriented surface $F$, if there exists an embedding $h : \Sigma \longrightarrow F$, then $h(\mathcal{D})$ is a link diagram in $F$. We call $h(\mathcal{D})$ a *link diagram realization* of $P = (\Sigma, \mathcal{D})$ in $F$. Figure 3 (c) and (d) are link diagram realizations of the ALDs illustrated in Figure 3 (a) and (b), respectively.



(a) (b)

abstractlinkdiagrams



(c) (d)

linkdiagramrealaizations

Figure 3

The *supporting genus* $g(P)$ of $P = (\Sigma, \mathcal{D})$ is the minimal genus of a closed oriented surface in which $\Sigma$ can be embedded.

**Lemma 8.** *Let $P = (\Sigma, \mathcal{D})$ be an ALD, which is the disjoint union of $m$ connected ALDs. Then*

$$g(P) = \frac{2m + c(\mathcal{D}) - \sharp\partial\Sigma}{2},$$

*where $c(\mathcal{D})$ is the number of crossings of $\mathcal{D}$, $\partial\Sigma$ is the boundary of the surface $\Sigma$ and $\sharp\partial\Sigma$ is the number of connected components of $\partial\Sigma$.*

Let $D$ be a virtual link diagram. Consider a link diagram realization $\mathcal{D}$ of $D$ in a closed oriented surface $F$ and take a regular neighborhood $N(\mathcal{D})$ of $\mathcal{D}$ in $F$. Then $(N(\mathcal{D}), \mathcal{D})$ is an ALD. It dose not depend on a link diagram realization. We call this the *ALD associated with* $D$, and denote it by $\phi(D)$. An easy method to obtain $\phi(D)$ is illustrated in Figure 4 (see [8] for details). For example, the ALDs illustrated in Figure 3 (a) and (b) are the ALDs associated with the virtual link diagrams in Figure 5 (a) and (b), respectively.

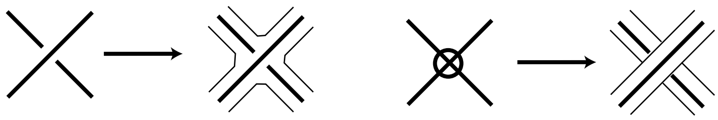


Figure 4

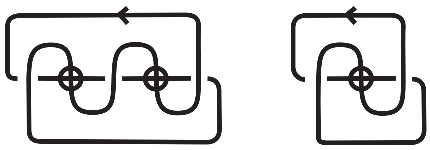


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Figure 5

**Lemma 9.** *Let $D$ be a virtual link diagram and let $\phi(D) = P = (\Sigma, \mathcal{D})$ be the ALD associated with $D$.*

(1) *$g(P) = g(D)$*
(2) *$P$ is connected if only if $D$ is connected.*

**Remark 10.** Let $P = (\Sigma, \mathcal{D})$ and $P' = (\Sigma', \mathcal{D}')$ be ALDs. We say that $P'$ is obtained from $P$ by an *abstract Reidemeister move* if there are embeddings $h : \Sigma \longrightarrow F$ and $h' : \Sigma' \longrightarrow F$ into a closed oriented surface $F$ such that the link diagram $h(\mathcal{D}')$ is obtained from $h(\mathcal{D})$ by a Reidemeister move in $F$. Two ALDs $P = (\Sigma, \mathcal{D})$ and $P' = (\Sigma', \mathcal{D}')$ are *equivalent* if there exists a finite sequence of ALDs, $P_0, P_1, \cdots, P_u$, with $P_0 = P$ and $P_u = P'$ such that $P_{i+1}$ is obtained from $P_i$ by an abstract Reidemeister move. An *abstract link* is such an equivalence class (cf. [2], [3], [8]). It is proved in [8] that two virtual link diagrams $D$ and $D'$ are equivalent if and only if the associated ALDs, $\phi(D)$ and $\phi(D')$, are equivalent; namely, the map

$$\phi : \{\text{virtual link diagrams}\} \longrightarrow \{\text{abstract link diagrams}\}$$

induces a bijection

$$\{\text{virtual links}\} \longrightarrow \{\text{abstract links}\}.$$

Let $P = (\Sigma, \mathcal{D})$ be an ALD. A crossing $p$ of $\mathcal{D}$ is *proper* if four distinct connected components of $\partial\Sigma$ pass through the neighborhood of $p$ as in Figure 6. When every crossing of $\mathcal{D}$ is proper, we say that $P$ is *proper*. Let $D$ be a virtual link diagram and $\phi(D) = (\Sigma, \mathcal{D})$ the ALD associated with $D$. A real crossing of $D$ is said to be *proper* if the corresponding crossing of $\mathcal{D}$ is proper. A virtual link diagram $D$ is said to be *proper* if each crossing of $D$ is proper (or equivalently if $\phi(D)$ is a proper ALD).

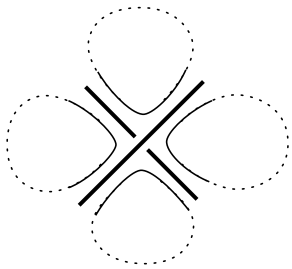
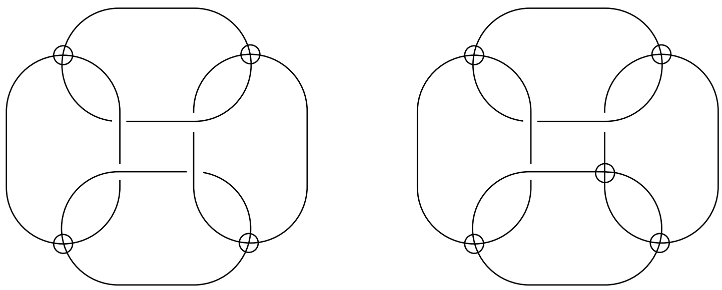


Figure 6

The left hand side of Figure 7 is a proper alternating virtual link diagram and the right hand side is a non-proper virtual link diagram. The right hand side is a v-alternating virtual link diagram obtained from the left diagram by virtualizing a real crossing.



properalternatinglinkdiagram v-alternatinglinkdiagram

Figure 7

## 4. Application

For non-zero integer $r_1, \cdots, r_s$, we denote by $K(r_1, \cdots, r_s)$ a virtual link diagram illustrated in Figure 8. The virtual link represented by this diagram is also denoted by $K(r_1, \cdots, r_s)$.

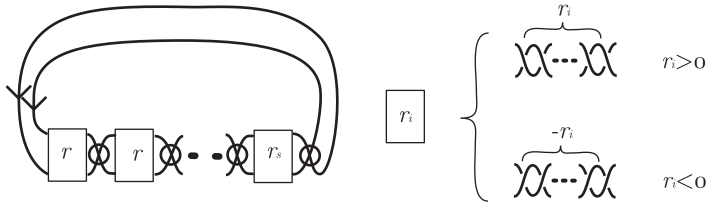



Figure 8

Kauffman proved that the $f$-polynomials of a virtual link diagram is invariant under the local moves as in Figure 9.

Using the moves in Figure 9 and generalized Reidemeister moves, we see that the $f$-polynomial of $K(r_1, \cdots, r_s)$ is equal to the $f$-polynomial of a virtual link

Figure 9

illustrated in Figure 10, where $r = r_1 + \cdots + r_s$. If $s$ is even, it is $(2, r)$-torus link or a trivial link. If $s$ is odd and $r \neq 0$, it is a v-alternating virtual link diagram satisfying the hypothesis of Corollary 6. Thus we have the following.

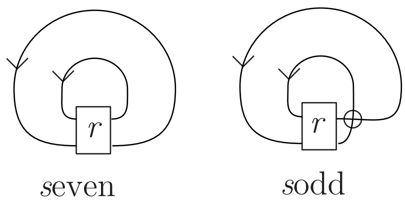



Figure 10

**Corollary 11.** (1) *If $s$ is odd and $r_1 + \cdots + r_s \neq 0$, then $K(r_1, \cdots .r_s)$ is not a classical link.*

(2) *If $s$ is odd, $r_1 + \cdots + r_s \neq 0$ and $s'$ is even, then $K(r_1, \cdots .r_s)$ and $K(r'_1, \cdots .r'_{s'})$ are not equivalent.*

**Theorem 12.** *For any positive integer $n$, there exists an infinite family of virtual link diagrams, $D(n, r)$ $(r = 0, 1, 2, \cdots)$, such that*

(1) *$D(n, r)$ is a proper alternating virtual link diagram,*
(2) *the supporting genus is $n$, and*
(3) *$c(D(n, r)) = 10n + r - 2$.*

*Proof.* A diagram $D(n, r)$ illustrated in Figure 11 satisfies the conditions. In the figure, the boxed $r$ stands for the $r$ right half twists. The supporting genus is $n$,

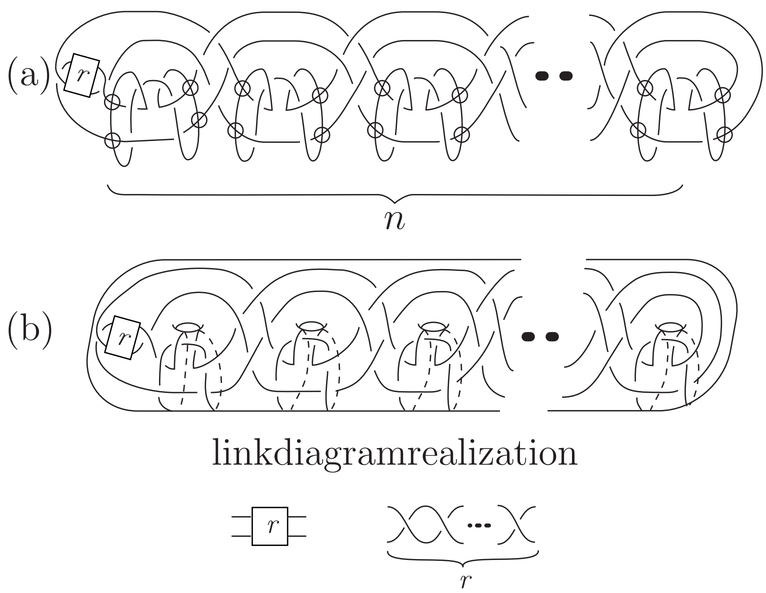



Figure 11

since it has a link diagram realization as in Figure 11(b) on a genus $n$ surface such that the complementary region consists of open disks. □

**Corollary 13.** *For any positive integer $N$, there are proper alternating virtual knot diagrams $D_1, \cdots, D_N$ with the same crossing number and the supporting genus of $D_k$ is $k$ ($k = 1, \cdots, N$).*

*Proof.* Let $D_k$ be the diagram $D(k, 10(N-k))$ introduced in Theorem 12. The crossing number of $D_k$ is $10N - 2$. □

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