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BRAIDS AND SINGULAR BRAIDS ON SURFACES

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ABSTRACT. This paper is a small survey on some recent developments on surface braid groups (denoted by $B_n(M)$), and on surface singular braid monoids (denoted by $SB_n(M)$). We show presentations for both, $B_n(M)$ and $SB_n(M)$, we explain the theory of Vassiliev invariants for braids on surfaces, we explain some combinatorial properties of the monoid $SB_n(M)$ (word problem, ...), and we present some subgroup $K_n(M)$ of $B_n(M)$ which is of importance in the study of braids on surfaces.

1. SURFACE BRAID GROUPS

Let M be a surface and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a collection of n distinct punctures in the interior of M. Define a *braid of M on n strings based at $\mathcal{P}$* to be a n-tuple $\beta = (b_1, \dots, b_n)$ of n disjoint smooth paths in $M \times [0,1]$, called the *strings* of β, such that

- the projection of $b_i(t)$ on the second coordinate is t, for all $t \in [0,1]$ and all $i \in \{1, \dots, n\}$;
- $b_i(0) = (P_i, 0)$ and $b_i(1) = (P_{\xi(i)}, 1)$, where ξ is a permutation of $\{1, \dots, n\}$, for all $i = 1, \dots, n$.

See Figure 1. The isotopy classes of braids based at $\mathcal{P}$ form a group, called the *braid group of M on n strings based at $\mathcal{P}$*, and denoted by $B_n(M) = B_n(M, \mathcal{P})$. Multiplication is by concatenation. Note that this group does not depend on $\mathcal{P}$, up to isomorphism, but only on the cardinality $n = |\mathcal{P}|$.

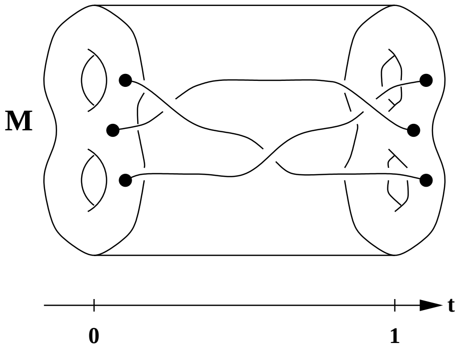

FIGURE 1. A braid of M.

The *Artin braid group B_n* is defined to be the braid group on n strings of the plane $\mathbb{E}^2$. This group was introduced by Artin in 1925 (see [1], [2]), and plays a prominent

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role in many disciplines. The natural extension to braid groups of topological spaces (and, in particular, of surfaces) was introduced by Zariski [27], Fox and Neuwirth [14] in terms of configuration spaces. Presentations for braid groups of closed surfaces have been calculated (see [27], [24], [17], and [5]), these groups are strongly related to mapping class groups (see [8]), but very few combinatorial properties of them are known. Recently, Irmak, Ivanov and McCarthy [20] have shown that all the automorphisms of $B_n(M)$ are geometric (*i.e.* are induced by diffeomorphisms of M), provided M is an oriented surface of genus $g \geq 2$, and $n \geq 3$. Another important result concerning these groups is a generalization of Markov's theorem which relates braids on surfaces to 3-dimensional manifolds (see [25]).

Let M be a closed oriented surface of genus $g \geq 1$. We represent the surface M by a polygon L of $4g$ sides identified as in Figure 2. Note that this identification has the effect of identifying all the vertices of the polygon to a single point of the surface M.

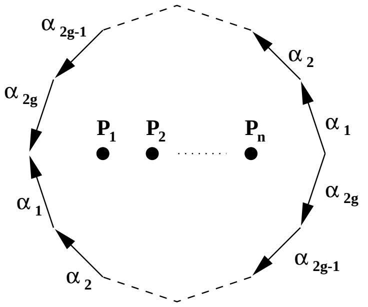

FIGURE 2. The polygon which represents M.

We consider the cylinder $L \times [0,1]$ and represent a braid $\beta = (b_1, \dots, b_n)$ as it is usually done for the open disk, that is, drawing in $L \times \{t\}$ the n points $b_1(t), \dots, b_n(t)$. Note that a string can cross a wall of the cylinder (see Figure 3). Our standard representation for a given braid β consists on looking at the above cylinder from the left hand side as in Figure 3. Now, the strings are represented by paths in the polygon L (which may cross sides), and, when two strings cross, we represent passing above the one that is first to reach the crossing point.

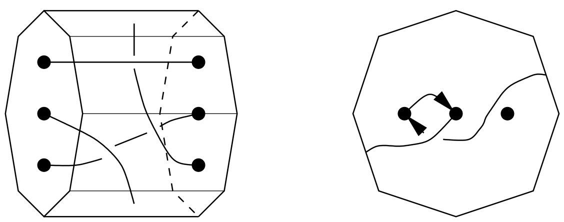

FIGURE 3. Two representations of a braid.

It is easily checked that the braids $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_{2g}$ represented in Figure 4 generate $B_n(M)$. A presentation for $B_n(M)$ with these generators has been calculated in [17], and the result is the following.

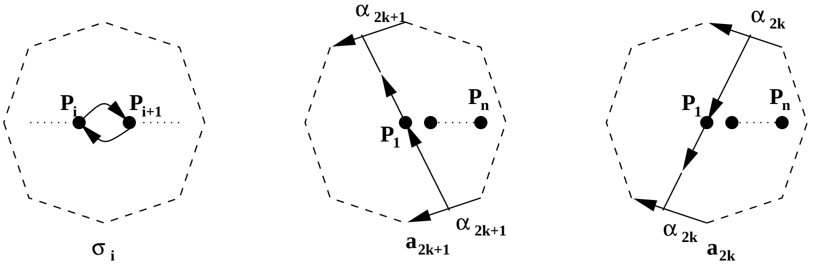

FIGURE 4. Generators of $B_n(M)$.

Theorem 1 (González-Meneses [17]**).** *The group $B_n(M)$ has a presentation with generators*

$$\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_{2g},$$

and relations

(R1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *if* $|i-j| \geq 2$,

(R2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *if* $1 \leq i \leq n-2$,

(R3) $a_1 \dots a_{2g} a_1^{-1} \dots a_{2g}^{-1} = \sigma_1 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_1$

(R4) $a_r A_{2\,s} = A_{2\,s} a_r$ *if* $1 \leq r, s \leq 2g$ *and* $r \neq s$,

(R5) $(a_1 \dots a_r)A_{2\,r} = \sigma_1^2 A_{2\,r}(a_1 \dots a_r)$ *if* $1 \leq r \leq 2g$,

(R6) $a_r\sigma_i = \sigma_i a_r$ *if* $1 \leq r \leq 2g$ *and* $2 \leq i \leq n-1$,

where

$$A_{2\,r} = \sigma_1^{-1}(a_1 \dots a_{r-1}a_{r+1}^{-1} \dots a_{2g}^{-1})\sigma_1^{-1}.$$

2. SINGULAR BRAID MONOIDS

In the same way as Artin braid groups have been extended to singular braid monoids (see [3], [10]), one can extend the braid group $B_n(M)$ to $SB_n(M)$, the monoid of singular braids.

A *singular braid of M on n strings based at $\mathcal{P}$* is defined to be a n-tuple $\beta = (b_1, \dots, b_n)$ of smooth paths in $M \times [0,1]$, called the *strings* of β, such that

- the projection of $b_i(t)$ on the second coordinate is t, for all $i \in \{1, \dots, n\}$ and all $t \in [0,1]$;
- $b_i(0) = (P_i, 0)$ and $b_i(1) = (P_{\xi(i)}, 1)$, where ξ is a permutation of $\{1, \dots, n\}$, for all $i \in \{1, \dots, n\}$;
- the strings of β intersect in finitely many double points, called *singular points*, and, at each singular point, the tangent lines to the two strings which intersect at the point span a plane.

See Figure 5. The isotopy classes of singular braids based at $\mathcal{P}$ form a monoid (and not a group), called the *singular braid monoid of M on n strings based at $\mathcal{P}$*, and denoted by $SB_n(M)$. It obviously contains the braid group $B_n(M)$.

Consider the singular braid τ_i, for $i = 1, \dots, n-1$, represented in Figure 6. Then one can easily verify that $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, a_1^{\pm 1}, \dots, a_{2g}^{\pm 1}, \tau_1, \dots, \tau_{n-1}$ generate $SB_n(M)$

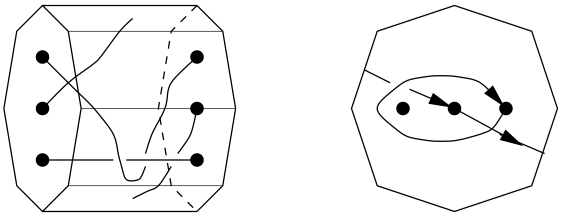

FIGURE 5. Two representations of a singular braid.

(as a monoid). A presentation for $SB_n(M)$ with these generators can be found in [18], and the result is the following.

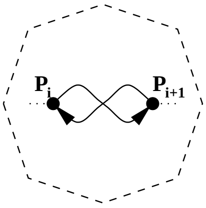

FIGURE 6. The singular braid τ_i.

Theorem 2 (González-Meneses [18]). *The monoid $SB_n(M)$ has a monoid presentation with generators*

$$\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, a_1^{\pm 1}, \dots, a_{2g}^{\pm 1}, \tau_1, \dots, \tau_{n-1}\,,$$

and relations

$$\begin{array}{lcl}
\textit{(R0)} & \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 & \textit{if } 1 \le i \le n-1\,, \\
 & a_k a_k^{-1} = a_k^{-1} a_k = 1 & \textit{if } 1 \le k \le 2g\,, \\
\textit{(R1)-(R6)} & \textit{Relations of } B_n(M) & \\
\textit{(R7)} & \sigma_i\tau_j = \tau_j\sigma_i & \textit{if } |i-j| \ge 2\,, \\
\textit{(R8)} & \tau_i\tau_j = \tau_j\tau_i & \textit{if } |i-j| \ge 2\,, \\
\textit{(R9)} & \sigma_i\tau_i = \tau_i\sigma_i & \textit{if } 1 \le i \le n-1\,, \\
\textit{(R10)} & \sigma_i\sigma_j\tau_i = \tau_j\sigma_i\sigma_j & \textit{if } |i-j| = 1\,, \\
\textit{(R11)} & (a_{i\,r}a_{i+1\,r})\tau_i(a_{i+1\,r}^{-1}a_{i\,r}^{-1}) = \tau_i & \textit{if } 1 \le i \le n-1 \textit{ and } 1 \le r \le 2g\,, \\
\textit{(R12)} & \tau_i a_{j\,r} = a_{j\,r}\tau_i & \textit{if } j \ne i, i+1 \textit{ and } 1 \le r \le 2g\,,
\end{array}$$

where

$$a_{i\,r} = \begin{cases} (\sigma_{i-1}^{-1} \dots \sigma_1^{-1}) a_r (\sigma_1^{-1} \dots \sigma_{i-1}^{-1}) & \textit{if } r \equiv 1\,(mod 2)\,, \\ (\sigma_{i-1} \dots \sigma_1) a_r (\sigma_1 \dots \sigma_{i-1}) & \textit{if } r \equiv 0\,(mod 2)\,. \end{cases}$$

3. DESINGULARIZATION AND VASSILIEV INVARIANTS

Define the *order* of a singular braid to be its number of singular points. Consider a singular braid β of order $d \ge 1$, and take a singular point P of β. We can slightly modify β in a small neighborhood of P in order to suppress the singular point. Following this modification, we obtain two singular braids of order $d-1$, denoted by β_+ and β_-, and called *resolutions of β at P*, as illustrated in Figure 7. Let $\mathbb{Z}[B_n(M)]$ denote the group algebra of $B_n(M)$. Then we define the *desingularization map* $\eta : SB_n(M) \to \mathbb{Z}[B_n(M)]$ by induction on the order of a singular braid, setting $\eta(\beta) = \beta$ if β is a non-singular braid, and $\eta(\beta) = \eta(\beta_+) - \eta(\beta_-)$ if β is a singular braid of order $d \geq 1$, and β_+, β_- are the resolutions of β at some singular point. One can easily verify that $\eta : SB_n(M) \to \mathbb{Z}[B_n(M)]$ is a well-defined multiplicative homomorphism. Actually, the desingularization map can also be defined as the multiplicative homomorphism $\eta : SB_n(M) \to \mathbb{Z}[B_n(M)]$ determined by

$$\begin{aligned} \eta(\sigma_i^{\pm 1}) &= \sigma_i^{\pm 1} && \text{for } 1 \leq i \leq n-1\,, \\ \eta(a_k^{\pm 1}) &= a_k^{\pm 1} && \text{for } 1 \leq k \leq 2g\,, \\ \eta(\tau_i) &= \sigma_i - \sigma_i^{-1} && \text{for } 1 \leq i \leq n-1\,. \end{aligned}$$

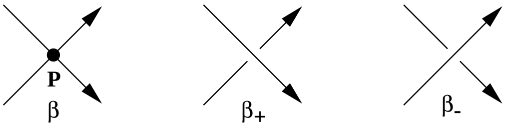

FIGURE 7. Resolutions at a singular point.

The following result answers some Birman's question (see [10]) in the context of braids on surfaces, and has been a quite popular problem in the subject.

Theorem 3 (Paris [23]**).** *The desingularization map $\eta : SB_n(M) \to \mathbb{Z}[B_n(M)]$ is injective.*

Let $S_dB_n(M)$ denote the set of isotopy classes of singular braids of order d. Following the same ideology as for knots, links, and other "knot-like" categories, we define a *Vassiliev invariant of order d* to be a homomorphism $v : \mathbb{Z}[B_n(M)] \to A$ of $\mathbb{Z}$-modules which vanishes on $\eta(S_{d+1}B_n(M))$.

Theorem 4 (González-Meneses, Paris [19]**).** *Vassilev invariants with values in $\mathbb{Z}$ separate braids on surfaces. Namely, given $\alpha, \beta \in B_n(M)$, $\alpha \neq \beta$, there exists a Vassiliev invariant $v : \mathbb{Z}[B_n(M)] \to \mathbb{Z}$ such that $v(\alpha) \neq v(\beta)$.*

This result is also known for Artin braids (see [4], [21], [22]), but is still a conjecture for knots.

One of the main preoccupations of the people working on Vassilev invariants is the definition and the study of the so-called universal invariants. In the case of braids on surfaces, a universal invariant has been constructed in [19] as follows.

For $d \in \mathbb{N}$, we denote by V_d the $\mathbb{Z}$-submodule of $\mathbb{Z}[B_n(M)]$ generated by $\eta(S_dB_n(M))$. It is easily checked that V_d is actually a two-sided ideal of $\mathbb{Z}[B_n(M)]$ and that the family $\{V_d\}_{d=0}^{+\infty}$ is a filtration of $\mathbb{Z}[B_n(M)]$, called the *Vassiliev filtration* of $\mathbb{Z}[B_n(M)]$.

A *chord diagram* is a diagram made of n vertical lines and of a finite number of horizontal segments, called *chords*, connecting the lines. A *M-labeled chord diagram* is a chord diagram such that each chord is labeled by an element of $\pi_1(M)$ (see Figure 8). Note that the set of labeled chord diagrams is equipped with a multiplication defined by concatenation. The free $\mathbb{Z}$-module generated by the chord diagrams is the free non-commutative $\mathbb{Z}$-algebra $\mathbb{Z}[t_{i\,j\,\gamma}]$ freely generated by the $t_{i\,j\,\gamma}$, where $i,j \in \{1,\dots,n\}$, $i \neq j$, $\gamma \in \pi_1(M)$, and where $t_{i\,j\,\gamma} = t_{j\,i\,\gamma^{-1}}$ (see Figure 8).

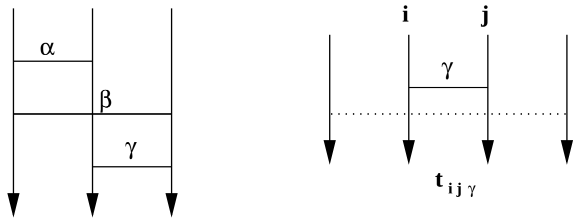

FIGURE 8. A M-labeled chord diagram and the generator $t_{i\,j\,\gamma}$.

We denote by $\mathcal{A}_n$ the quotient of the free $\mathbb{Z}$-algebra $\mathbb{Z}[t_{i\,j\,\gamma}]$ by the relations:

- $[t_{i\,j\,\gamma}, t_{k\,l\,\delta}] = 0$, for all distinct $i,j,k,l \in \{1,\dots,n\}$, and all $\gamma,\delta \in \pi_1(M)$,
- $[t_{i\,j\,\gamma}, t_{j\,k\,\delta} + t_{i\,k\,\gamma\delta}] = 0$, for all distinct $i,j,k \in \{1,\dots,n\}$, and all $\gamma,\delta \in \pi_1(M)$,

and we denote by $\hat{\mathcal{A}}_n$ the natural completion of $\mathcal{A}_n$. The symmetric group Sym_n acts on $\pi_1(M)^n$ by permuting the coordinates, so we can consider the induced semidirect product $H_n = \pi_1(M)^n \rtimes \mathrm{Sym}_n$. In addition, it is straightforward to show that H_n acts on $\hat{\mathcal{A}}_n$, and therefore defines a semidirect product $\hat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$. The $\mathbb{Z}$-algebra $\hat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$ carries the filtration induced by that of $\hat{\mathcal{A}}_n$, so its associated graded algebra is $\mathcal{A}_n \rtimes \mathbb{Z}[H_n]$. We also have $\mathrm{gr}_V\mathbb{Z}[B_n(M)] = \oplus_{d=0}^{+\infty}(V_d/V_{d+1})$.

Theorem 5 (González-Meneses, Paris [19]). *There exists a homomorphism* $Z : \mathbb{Z}[B_n(M)] \to \hat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$ *of* $\mathbb{Z}$*-modules such that the associated graded map*

$$\mathrm{gr}Z : \mathrm{gr}_V\mathbb{Z}[B_n(M)] \to \mathcal{A}_n \rtimes \mathbb{Z}[H_n]$$

is an isomorphism.

The homomorphism Z of the above theorem is called a *universal invariant* because of the following.

Corollary 6. *Every Vassiliev invariant of* $B_n(M)$ *factors through* Z *in a unique way. Namely, if* $v : \mathbb{Z}[B_n(M)] \to A$ *is a Vassiliev invariant, then there exists a unique homomorphism* $\hat{v} : \hat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n] \to A$ *such that* $v = \hat{v} \circ Z$*.*

Remark. By [7], the map Z cannot be functorial (*i.e.* an algebra homomorphism).

4. COMBINATORICS OF $SB_n(M)$

Let Γ be a graph (with no loop and no multiple edge), let X be the set of vertices of Γ, and let $E = E(\Gamma)$ be the set of edges of Γ. Define the *graph monoid* of Γ to be the monoid $\mathcal{M}(\Gamma)$ given by the monoid presentation

$$\mathcal{M}(\Gamma) = \langle X \mid xy = yx \text{ if } \{x,y\} \in E(\Gamma)\rangle^+ .$$

Graph monoids are also known as *free partially commutative monoids* or as *right-angled Artin monoids.* They were first introduced by Cartier and Foata [11] to study combinatorial problems on rearrangements of words, and, since then, have been extensively studied by both, computer scientists and mathematicians.

Now, consider the graph $\hat{\Omega}$ defined as follows.

- $\hat{\Upsilon} = \{\alpha\tau_i\alpha^{-1}; \alpha \in B_n(M) \text{ and } 1 \le i \le n-1\}$ is the set of vertices of $\hat{\Omega}$;
- $\{\hat{u}, \hat{v}\}$ is an edge of $\hat{\Omega}$ if $\hat{u}\hat{v} = \hat{v}\hat{u}$ in $SB_n(M)$.

The following result is a straightforward consequence of the presentation of $SB_n(M)$ given in Theorem 2.

Proposition 7 (Paris [23]**).** *We have* $SB_n(M) = \mathcal{M}(\hat{\Omega}) \rtimes B_n(M)$.

Corollary 8. $SB_n(M)$ *embeds in a group.*

Proof. Let $\mathcal{G}(\hat{\Omega})$ be the group presented by $\mathcal{G}(\hat{\Omega}) = \langle \hat{\Upsilon} \mid \hat{u}\hat{v} = \hat{v}\hat{u} \text{ if } \{\hat{u}, \hat{v}\} \in E(\hat{\Omega})\rangle$. Then $\mathcal{M}(\hat{\Omega})$ embeds in $\mathcal{G}(\hat{\Omega})$ (see [12]), thus $SB_n(M) = \mathcal{M}(\hat{\Omega}) \rtimes B_n(M)$ embeds in $\mathcal{G}(\hat{\Omega}) \rtimes B_n(M)$. ☐

A solution to the word problem for $B_n(M)$ can be found in [17]. Observe also that a given element of a graph monoid $\mathcal{M}(\Gamma)$ has finitely many representatives and that these representatives can be easily listed. Other solutions to the word problem for $\mathcal{M}(\Gamma)$ can be found in [11], [26], and [12]. By Proposition 7 and by the above observations, in order to solve the word problem in $SB_n(M)$, it suffices to find an algorithm which decides whether two elements $\alpha\tau_i\alpha^{-1}$ and $\beta\tau_j\beta^{-1}$ of $\hat{\Upsilon}$ are equal, and, if not, whether they commute or not. Such an algorithm can be easily derived from Proposition 9 below together with González-Meneses' solution to the word problem for $B_n(M)$.

Define the graph Ω as follows.

- $\Upsilon = \{\alpha\sigma_i\alpha^{-1}; \alpha \in B_n(M) \text{ and } 1 \le i \le n-1\}$ is the set of vertices of Ω;
- $\{u, v\}$ is an edge of Ω if $uv = vu$ in $B_n(M)$.

Proposition 9 (Paris [23]**).** *There exists an isomorphism* $\varphi : \hat{\Omega} \to \Omega$ *which sends* $\alpha\tau_i\alpha^{-1}$ *to* $\alpha\sigma_i\alpha^{-1}$ *for all* $\alpha \in B_n(M)$ *and all* $1 \le i \le n-1$.

In [13], Fenn, Rolfsen and Zhu proved that the centralizer in the singular Artin braid monoid of a standard singular generator τ_i is equal to the centralizer of any non-zero power of τ_i, and that this centralizer coincides with the centralizer of any non-zero power of σ_i. This property, which we like to call the *FRZ property*, is of importance in the study of singular Artin braids. The FRZ property has been extended to the singular braids on surfaces by Bellingeri [6], and it is the key of the proof of Proposition 9.

Proposition 10 (Bellingeri [6]**).** *Let* $\alpha \in SB_n(M)$ *and* $1 \le i, j \le n-1$. *Then the following are equivalent.*

(1) $\alpha\sigma_i = \sigma_j\alpha$.

(2) There exists $k \in \mathbb{Z} \setminus \{0\}$ *such that* $\alpha\sigma_i^k = \sigma_j^k\alpha$.

(3) $\alpha\tau_i = \tau_j\alpha$.

(4) There exists $k \in \mathbb{N} \setminus \{0\}$ *such that* $\alpha\tau_i^k = \tau_j^k\alpha$.

5. THE GROUP $K_n(M)$

One of the main ingredients in the study of the Artin braid groups is the fact that the pure braid group PB_n can be decomposed as $PB_n = F_{n-1} \rtimes PB_{n-1}$, where F_{n-1} is a free group freely generated by some set $\{T_{1\,2}, \dots, T_{1\,n}\}$, and that the conjugacy class in F_{n-1} of each $T_{1\,j}$ is invariant by the action of PB_{n-1}. In particular, PB_{n-1} acts trivially on the abelianization of F_{n-1}. These facts are not true anymore for pure braid groups on closed surfaces. If M is a closed surface, then we do have an exact sequence $1 \to R_n(M) \to PB_n(M) \to PB_{n-1}(M) \to 1$, where $R_n(M)$ is a free group, but, in general, this exact sequence does not split (see [16]), and, moreover, the action of PB_{n-1} on the abelianization of $R_n(M)$ is not trivial. In order to palliate these difficulties, we should replace the pure braid group by some subgroup $K_n(M)$ of $B_n(M)$ defined as follows.

Define the homomorphism $\phi : PB_n(M) \to \pi_1(M)^n$ as follows. Let $\beta = (b_1, \dots, b_n)$ be a pure braid. For $i = 1, \dots, n$, let $\bar{b}_i$ be the projection of b_i on the first coordinate, and let μ_i be the element of $\pi_1(M) = \pi_1(M, P_i)$ represented by $\bar{b}_i$. Then $\phi(\beta) = (\mu_1, \dots, \mu_n)$. The group $K_n(M)$ is defined to be the kernel of ϕ.

An alternative definition of $K_n(M)$ is the following. Choose a disk $\mathbb{D}$ embedded in M and containing $\mathcal{P} = \{P_1, \dots, P_n\}$. Then the inclusion $\mathbb{D} \subset M$ determines a homomorphism $\iota : PB_n(\mathbb{D}) \to PB_n(M)$. By [15], this homomorphism is injective and $K_n(M)$ is the normal closure of $\iota(PB_n(\mathbb{D}))$ in $PB_n(M)$.

Let $\rho : PB_n(M) \to PB_{n-1}(M)$ be the epimorphism which sends a pure braid $\beta = (b_1, \dots, b_n)$ to $(b_1, \dots, b_{n-1})$. Let $\mathcal{P}_{n-1} = \{P_1, P_2, \dots, P_{n-1}\}$. By [14], the kernel of ρ is $R_n(M) = \pi_1(M \setminus \mathcal{P}_{n-1}) = \pi_1(M \setminus \mathcal{P}_{n-1}, P_n)$ which, clearly, is a free group. Let $F_n(M) = K_n(M) \cap R_n(M)$. Then $F_n(M)$ is an (infinitely generated) free group and we have the exact sequence

$$1 \to F_n(M) \longrightarrow K_n(M) \xrightarrow{\rho} K_{n-1}(M) \to 1\,.$$

Theorem 11 (González-Meneses, Paris [19], [23]**).** *(1) The epimorphism $\rho : K_n(M) \to K_{n-1}(M)$ admits a section. In particular, $K_n(M)$ is of the form $K_n(M) = F_n(M) \rtimes K_{n-1}(M)$.*

(2) There is a basis $\mathcal{B}$ of $F_n(M)$ such that the conjugacy class in $F_n(M)$ of every element of $\mathcal{B}$ is invariant by the action of $K_{n-1}(M)$. In particular, $K_{n-1}(M)$ acts trivially on the abelianization of $F_n(M)$.

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