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THE REDUCIBILITIES OF HEEGAARD SPLITTINGS

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Abstract. In this talk, we shall use Scharlemann-Thompson's powerful ideas of thin decompositions of Heegaard splittings to give a new version to Casson-Gordon's theorem and Haken's lemma on Heegaard splittings. We shall also introduce some new results on the stabilizations of reducible Heegaard splittings and amalgamations of Heegaard splittings.

1. Preliminaries

1.1 Fundamental facts on 3-manifolds

Let M be a 3-manifold and S be a 2-sphere in M. If S bounds a 3-ball in M, we say that S is *trivial* in M; otherwise, we say S is *essential* in M. If M contains an essential 2-sphere, we say that M is *reducible*; otherwise, M is irreducible.

Suppose F is a surface in M, F is properly embedded in M or $F \subset \partial M$. If one of the following happens:

(1) F is a disk in ∂M, or F is a proper disk and cobounds with a disk in ∂M a 3-ball in M; or

(2) F is a trivial 2-sphere in M; or

(3) there is a disk D in M so that $D \cap F = \partial D$ and ∂D bounds no disk in F,

then we say that F is *compressible* in M. In the third case, we often say that D is a compression disk of F in M.

If F is not compressible in M, we say that F is *incompressible* in M.

Let F be a properly embedded surface in 3-manifold M. If there exists a disk $\Delta \subset M$ such that $\Delta \cap F = \alpha$ is an essential arc in F, $\Delta \cap \partial M = \beta$ is an arc in ∂M, $\partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial\Delta$, we say F is *∂-compressible in M*, and Δ is a ∂-compression disk for F in M. If F is not ∂-compressible in M, we say that F is ∂-incompressible in M.

A *compression body* C is a 3-manifold obtained by adding 2-handles to $S \times I$, where S is a connected closed surface, along a collection of pairwise disjoint simple closed curves on $S \times \{0\}$, then capping off any resulting 2-sphere boundary components with 3-balls. Denote by ∂_+C the surface $S \times \{1\}$ in ∂C , and $\partial_-C = \partial C - \partial_+C$. When $\partial_-C = \emptyset$, C is a handlebody. When $C = S \times I$, C is a trivial compression body.

There is a dual picture for a compression body C. C can be regarded as a 3-manifold obtained by attaching some 1-handles to $\partial_-C \times I$ from one side of the product if $\partial_-C \neq \emptyset$, or to a 0-handle if $\partial_-C = \emptyset$. Each of the cores of the 1-handles is called a spine of the compression body.

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A *complete disk system* B for a compression body C is a disjoint union of disks $(D, \partial D) \subset (C, \partial_+ C)$ such that the manifold obtained by cutting C along B is homeomorphic to $\partial_- C \times I$ if $\partial_- C \neq \emptyset$, or a 3-ball if $\partial_- C = \emptyset$.

Note that the cores of the 2-handles in constructing the compression body C can be vertically extended to C through $S \times I$, which contains a complete disk system for C. If we view C from the dual picture, the collection of co-cores of the 1-handles is a complete disk system for C.

Let M be a 3-manifold such that ∂M has no 2-sphere components. A Heegaard splitting of M is a pair (V, W), where V, W are compression bodies such that $V \cup W = M$, and $V \cap W = \partial_+ V = \partial_+ W = F$. F is called a Heegaard surface in M. The splitting is often denoted as $V \cup_F W$, and the genus of F is called the genus of the Heegaard splitting.

Let $V \cup_F W$ be a Heegaard splitting for M. $V \cup_F W$ is *reducible* (or *weakly reducible*) if there exist essential disks $D \subset V$ and $E \subset W$ with $\partial D = \partial E$ (or $\partial D \cap \partial E = \emptyset$). If $V \cup_F W$ is not reducible, we say it is irreducible; and if $V \cup_F W$ is not weakly reducible, we say it is strongly irreducible. $V \cup_F W$ is *stabilized* if there are two properly embedding disks $D_1 \subset V$ and $D_2 \subset W$ such that D_1 intersects D_2 in only one point. It is easy to see that $V \cup_F W$ is *reducible*if $V \cup_F W$ is *stabilized.*

Clearly, a reducible Heegaard splitting is weakly reducible.

It is easy to see that a compression body is irreducible, and for a non-trivial compression body C, $\partial_+ C$ is compressible in C, and if $\partial_- C \neq \emptyset$, $\partial_- C$ is incompressible in C.

The following proposition can be proved by using a complete disk system, we omit the proof (cf. [8, Lemma 2.3]).

Proposition 1.1 *Let F be a connected incompressible surface properly embedded in a compression body C. Then*

1) either F is a closed surface parallel to a component of $\partial_- C$; or

2) F is ∂-compressible in C; or

3) F is a (essential) disk with $\partial F \subset \partial_+ C$; or

4) F is an annulus with one component of ∂F lying in $\partial_+ C$ and the other in $\partial_- C$.

1.2 Thin decompositions from Heegaard splittings

In this part we review Scharlemann-Thompson's idea of thin decompositions from Heegaard splittings.

Let $V \cup_F W$ be an irreducible Heegaard splitting for a 3-manifold M. The compression body V is obtained from $\partial_- V \times I$ (or a 3-ball if $\partial_- V = \emptyset$) by attaching some 1-handles $\{h_i^1\}$ to one side of $\partial_- V \times I$ (or the 3-ball), and W is obtained from $\partial_+ W \times I$ by attaching some 2-handles $\{h_j^2\}$ then capping off any resulting 2-spheres by 3-handles $\{b_k\}$. Thus

$$M = X \cup \{h_i^1\} \cup \{h_j^2\} \cup \{b_k\},$$

where $X = \partial_- V \times I$ if $\partial_- V \neq \emptyset$ or X is a 0-handle if V is a handlebody. This describes the Heegaard splitting as a standard handle decomposition of M with cobordism between $\partial_- V$ and $\partial_- W$.

For a Heegaard splitting $V \cup_F W$ for M, one can begin with the handle structure determined by $V \cup_F W$ and rearrange the order of the 1-, 2- and 3-handles as follows:

$$M = X \cup N_1 \cup T_1 \cup Y_1 \cup N_2 \cup T_2 \cup Y_2 \cup \cdots \cup N_k \cup T_k \cup Y_k,$$

where $X = \partial_- V \times I$ if $\partial_- V \neq \emptyset$ or X is a 0-handle if V is a handlebody, N_i is a collection of some 1-handles in $\{h_i^1\}$, T_i is a collection of some 2-handles in $\{h_j^2\}$ and Y_i is a collection of 3-handles in $\{b_k\}$. Let F_i, $1 \leq i \leq k$ be the surface obtained from $\partial(X \cup N_1 \cup T_1 \cup Y_1 \cup N_2 \cup T_2 \cup Y_2 \cup \cdots \cup N_i)$. Let $S_i = \partial(X \cup N_1 \cup T_1 \cup Y_1 \cup N_2 \cup T_2 \cup Y_2 \cup \cdots \cup T_i \cup Y_i)$, $1 \leq i \leq k-1$ $\partial(X \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup T_i)$. Let $M_1 = M_0 \cup N_1 \cup T_1 \cup Y_1$, where $M_0 = \partial_- V \times [0,1]$ if $\partial_- V \neq \emptyset$ or a 3-ball if V is a handlebody, and $M_i = (\text{collar of } S_{i-1}) \cup N_i \cup T_i \cup Y_i$. Now if each component of S_i is not a 2-sphere, then M_i is divided by a copy of F_i into two compression bodies V_i and W_i, so $V_i \cup_{F_i} W_i$ is a Heegaard splitting for M_i.Thus

$$M = M_1 \cup_{S_1} M_2 \cup_{S_2} \cdots \cup_{S_{k-1}} M_k, \tag{1}$$

which is called a decomposition from $V \cup_F W$.

Lemma 1.2. If one component of S_i, for some i, is a 2-sphere, then the Heegaard splitting $V \cup_F W$ is reducible.

Proof. Suppose, for some i, one component of S_i is a 2-sphere, say P. By definitions, in V, there are some two handles attached to P. Similarly, in W, there are some 1-handles attached to P. We denote by B_1 the collection of attached disks in V and B_2 the collection of attached disks in W. Then there is a simple closed curve α in P which separates B_1 from B_2. That means that α is an essential simple closed curve in F which bounds disks $D_1 \subset V$ and $D_2 \subset W$. Thus $V \cup_F W$ is reducible.

Definition 1.3 For a closed connected orientable surface F, define the complexity $c(F) = 2g(F) - 1$, where $g(F)$ denotes the genus of F. Define $c(S^2) = 0$. For F not necessarily connected define $c(F) = \sum\{c(F')|F' \text{ a component of } F\}$.

Let $M = V \cup_F W$ has a decomposition as in (1). Define the *width* of the decomposition to be the set of integers $\{c(F_i)|1 \leq i \leq k\}$. Arrange each such set in monotonically non-increasing order, then compare the ordered sets lexicographically. Define the *width* $w(V \cup_F W)$ of the splitting $V \cup_F W$ to be the minimal width over all decompositions coming from $V \cup_F W$. A decomposition from $V \cup_F W$ is thin if the width of the decomposition is the width of $V \cup_F W$.

From the definition and Lemma 2.2, one can immediately get

Proposition 1.4 In a thin decomposition coming from an irreducible Heegaard splitting $V \cup_F W$ of M as in (1), $V_i \cup_{F_i} W_i$ is strongly irreducible for each i.

Remark 1.5 For a thin decomposition coming from a Heegaard splitting $V \cup_F W$ of M as in (1), it is not necessarily a thin decomposition of M in Scharlemann-Thompson's version, since in our version, a 2-sphere component of an S_i could be inessential in M.

2. The reducibilities of Heegaard splittings

It is well known that Haken's Lemma on Heegaard splittings ([1], Haken, 1968), Jaco's Lemma on handle addition ([2], Jaco, 1984) and Casson-Gordon's Theorem on reducible Heegaard splittings ([3], Casson-Gordon, 1987) play essential roles in the modern study of three manifold topology from the combinatorial point of view, and they have been applied in dealing many problems related to incompressible surfaces, Heegaard splitting, Dehn surgery, and related topics, and great success has been achieved.

In this present note, we give a new proof to the above theorems by Scharlemann-Thompson's idea of thin decomposition for 3-manifolds.

The following is the so called "Nested lemma", which is due to Boileau-Otal [6] and reproved by Scharlemann-Thompson[7]. We note that Scharlemann-Thompson's proof depends on the disk version of Haken's Lemma. We will describe an elementary proof which is independent of the disk version of Haken's Lemma. It will play an important role in our further discussion.

Lemma 2.1 *Suppose that $V \cup_F W$ is a strongly irreducible Heegaard splitting of a 3-manifold M, and D is a disk in M transverse to F with ∂D essential in F. Then ∂D also bounds a disk in V or W.*

Proof. If $F \cap int(D) = \emptyset$, there is nothing to prove. Note that each component of $F \cap int(D)$ is a circle. If $F \cap int(D)$ has a component which is trivial on F, let α be such a component so that α bounds a disk $\Delta' \subset F$ whose interior is disjoint from D. α bounds a disk Δ in D. Let $D' = (D - \Delta) \cup \Delta'$. Then $\partial D' = \partial D$, and after a small isotopy, $|F \cap int(D')| < |F \cap int(D)|$. Similarly, if $D \cap V$ (or $D \cap W$) is compressible in V (or W), we can again reduce $|F \cap int(D)|$. Thus we may assume that each component of $F \cap int(D)$ is essential in F, and $D \cap V$ ($D \cap W$, resp) is incompressible in V (W, resp). By the strong irreducibility of F, we may further assume all disk components of $D - F$ are lying, for example, in W.

We first prove the lemma in a special case:

Claim. Assume that every component of $F \cap int(D)$ is innermost in D, i.e. it bounds a sub-disk of D in W, then ∂D also bounds a disk in W.

Denote $P = D \cap V$. Then P is a planar surface with $\partial P \subset F$, and ∂P has at least two boundary components, one is ∂D, all the others bound essential disks in W.

We induct on $|\partial P|$. First consider $|\partial P| = 2$. P is an annulus. One component of ∂P is ∂D, the other one is denoted by α. Since P is incompressible in V, by Proposition 1, it is ∂-compressible in V. Let Δ be a ∂-compression disk of P in V, $\beta = \Delta \cap F, \gamma = \Delta \cap P$, and $\beta \cup \gamma = \partial\Delta$. β connects the distinct components of P. ∂-compress P along Λ in V to get a disk E. Note that α and ∂E can be made, by isotopy on F, to be disjoint, and α bounds an essential disk in W, the strong irreducibility of F implies that E is a trivial disk in V. Thus ∂D and α are parallel on F, so ∂D also bounds a disk in W.

Now assume that it holds for all disks D' in M satisfying the condition in the claim with $|D' \cap F| \leq m$ ($m \geq 2$). Let D be such a disk, $P = D \cap V$, and

$\partial P = \partial D \cup \{\alpha_i, 1 \leq i \leq m\}$. Again, P is ∂-compressible in V. Let Δ be a ∂-compression disk of P in V, $\beta = \Delta \cap F, \gamma = \Delta \cap P$, and $\beta \cup \gamma = \partial \Delta$. There are four subcases:

1) β connects ∂D. ∂-compress P along Δ to get P_1 and P_2, then ∂D gets into c_1 and c_2. By inductive assumption, c_i bounds a disk in W, $i = 1, 2$, and ∂D is a band sum of c_1 and c_2, so ∂D bounds a disk in W.

2) β connects α_1, say. ∂-compress P along Δ to get P_1 and P_2, then α_1 gets into α_1' and α_1''. Assume $\partial D \subset \partial P_1$. By inductive assumption, α_1'' bounds a disk in W. Note that α_1 bounds a disk in W, so α_1' also bounds a disk in W. By inductive assumption again, ∂D bounds a disk in W.

3) β connects ∂D and α_1 (say). ∂-compress P along Δ to get P'. Let α_1' be the band sum of ∂D and α_1 along β. By inductive assumption, α_1' bounds a disk in W. Note that α_1 bounds a disk in W, so ∂D also bounds a disk in W.

4) β connects α_1 and α_2 (say). ∂-compress P along Δ to get P'. Let α_1' be the band sum of α_1 and α_2 along β. Then α_1' bounds a disk in W. By inductive assumption, ∂D also bounds a disk in W.

This completes the proof of Claim.

For general case, let α be a component of $F \cap int(D)$ so that for the sub-disk $E \subset D$, each component of $F \cap int(E)$ is innermost in E. By claim, ∂E bounds a disk Σ in W. Since $D \cap W$ is incompressible in W, as before, we may assume $int(D \cap W) \cap int(\Sigma) = \emptyset$. Let $D' = (D - E) \cup \Sigma$. Then $\partial D' = \partial D$, and $|F \cap int(D')| < |F \cap int(D)|$. A finite induction on $|F \cap int(D)|$ finishes the proof.

A direct consequence of The nested Lemma is the following

Corollary 2.2 *Let M be a 3-manifold with a strongly irreducible Heegaard splitting. Then M is irreducible.*

Proof. Let $V \cup_F W$ be a strongly irreducible Heegaard splitting for M. If M is reducible, there exist essential spheres in M. For an essential sphere S in M, we can isotope S in M so that each component of $S \cap W$, say, is an essential disk in W. Choose an essential sphere, still denoted by S, in M so that $|S \cap F|$, the number of the disks in $S \cap W$, is minimal over all possible essential spheres in M. Since a compression body is irreducible, $S \cap F \neq \emptyset$. By the strong irreducibility of the splitting, $|S \cap F| > 1$. Let $P = S \cap V$. As before, P is a connected incompressible planar surface in V. By proposition 2.1, P is ∂-compressible in V. Let Δ be a ∂-compression disk of P in W, and $\alpha = \Delta \cap F$. If α connects distinct components of ∂P, then after ∂-compressing P along Δ and an isotopy, we can move S to S' in M so that $S' \cap W$ still consists of disks and $|S' \cap F| < |S \cap F|$, contradicting to our choice of S. If α connects the same component c of ∂P, then after ∂-compressing P along Δ and an isotopy, P gets into P_1 and P_2, c gets into c_1 and c_2. one of c_1 and c_2, say c_1, is essential in F, and each bounds a disk in M. By the nested lemma, c_1 bounds a disk in V or W. c_1 cannot bound a disk in V, because $V \cup_F W$ is strongly irreducible. So c_1 bounds a disk in W, thus c_2 also bounds a disk in W. Capping off P_1 and P_2 by disks in W, we get two spheres S' and S'', and at least one of them is essential in M. Note $|S' \cap F|, |S'' \cap F| < |S \cap F|$, again a contradiction.

Corollary 2.3 *Let M be a 3-manifold with a strongly irreducible non-trivial Heegaard splitting. Then ∂M is incompressible in M.*

Proof. Let $V \cup_F W$ be a strongly irreducible Heegaard splitting for M. Suppose that ∂M is compressible in M. Then there exists a compression disk of ∂M in M. Choose such a disk D in M with $\partial D \subset \partial_- V$, say, so that $D \cap F$ has minimal components over all possible choices. Since the splitting is non-trivial, $D \cap F \neq \emptyset$. By a similar argument use in the proof of the nested lemma, one can show that $D \cap F$ consists of a single essential circle. Thus $A = D \cap V$ is an essential annulus in V. V is not a trivial compression body, by an innermost circle and outermost arc argument, one can show that there exists a complete disk system $\mathcal{E}$ for V with $\mathcal{E} \cap A = \emptyset$. Thus if we denote $D' = D \cap W$, $\partial D' \cap \partial \mathcal{E} = \emptyset$, contradicting to the strong irreducibility of $V \cup_F W$.

Corollary 2.4 In a thin decomposition coming from a Heegaard splitting $V \cup_F W$ of M as in (1), for each component S_i' of S_i with positive genus, S_i' is incompressible in both M_i and M_{i+1} (therefore in M), $1 \leq i \leq k-1$.

Proof. This is a direct consequence of Proposition 2.3 and Corollary 3.3.

Corollary 2.5 (Haken's lemma in disk version) *Let $V \cup_F W$ be a Heegaard splitting for M. There is a thin decomposition from $V \cup_F W$ as in (1). Assume that ∂M is compressible in M, and D' is a compression disk of ∂M in M with $\partial D' \subset \partial_- V$. Then V_1 is a trivial compression body. In particular, there exists a compression disk D of ∂M in M such that D intersects F in a single circle.*

Proof. Let D a compression disk of $\partial_- V$ in M. For a component of S_1, either it is a 2-sphere or it is incompressible in M, by an innermost circle argument, we may assume that $D \cap S_1 = \emptyset$. Thus D is totally contained in $M_1 = V_1 \cap_{F_1} W_1$. The strong irreducibility of $V_1 \cup_{F_1} W_1$ implies that V_1 is trivial. Clearly, D intersects F in a single circle.

Corollary 2.6 (Haken's Lemma) *Let M be a reducible 3-manifold, $V \cup_F W$ a Heegaard splitting for M. Then there exists an essential 2-sphere S in M so that S intersects F in a single circle.*

Proof. Let $(V_1 \cup_{F_1} W_1) \cup_{S_1} \cdots \cup_{S_{k-1}} (V_k \cup_{F_k} W_k)$ be a thin decomposition coming from $V \cup_S W$. If each component of S_i is not a 2-sphere for $2 \leq i \leq k-1$, then by Corollaries 3.2 and 3.4, M is irreducible. If, for some i, one component of S_i is a 2-sphere, then, by Lemma 2.2, $V \cup_F W$ is reducible.

Corollary 2.7 (Casson-Gordon's Theorem) *Let $V \cup_F W$ is a weakly reducible Heegaard splitting for 3-manifold M. Then either $V \cup_F W$ is reducible, or M contains a closed incompressible surface of positive genus.*

Proof. Let $(V_1 \cup_{F_1} W_1) \cup_{S_1} \cdots \cup_{S_{k-1}} (V_k \cup_{F_k} W_k)$ be a thin decomposition coming from $V \cup_S W$. Since $V \cup_F W$ is weakly reducible, by proposition, $k \geq 2$. If a component of S_1 is a 2-sphere, then $V \cup_F W$ is reducible. Otherwise, by corollary 3.4, S_1 is incompressible in M with positive genus.

Corollary 2.8 (Jaco's Lemma) *Let M be an irreducible 3-manifold with compressible boundary, J a simple closed curve in ∂M. Suppose $\partial M - J$ is incompressible in M. Let M_J be the manifold obtained by attaching a 2-handle to M along J. Then either ∂M_J is incompressible in M_J, or M is a solid torus and J is a longitude of M.*

Proof. We only consider the case that M is not a compression body. The case that M is a compression body is similar. Let V be a maximal compression body properly embedded in M, i.e., $F' = \partial_- V$ is a closed surface in M which cuts M into V and V' and F' is incompressible in V'. $M_j = (V' \cup_{F'} V)_J = V' \cup_{F'} V_J$. Let F be a parallel copy of $\partial_+ V$ in the interior of V. Then F describes a Heegaard splitting for V_J. By assumption, the Heegaard splitting is nontrivial and strongly irreducible. From Corollary 3.2 and 3.3, V_J is irreducible and ∂-irreducible. In particular, F' is incompressible in V_J. Thus M_J is irreducible and ∂M_J is incompressible in M_J.

The above arguments are given by Lei and Qiu.

3. Stabilizations of Heegaard splittings

1. Some famous results

Theorem 1[Wa]. Each unstabilized Heegaard splittings is standard.

Theorem 2. Two Heegaard splittings of any closed manifold M can be unstabilized to a Heegaard splitting of M.

2. Tunnel numbers of knots

Let K be a knot in a manifold, and M_K be the complement of K in M. Now the number $T(K) = Min\{g(F) - 1 \mid M = V \cup_F W\}$ is called the tunnel number of K. The foundmental results on this problem are the following:

1) It is possible that $T(K_1 \sharp K_2) < T(k_1) + T(K_2)$. (Morimoto and others)

2) ***Theorem 2.*** $T(K_1 \sharp \ldots \sharp K_n) \geq n$. (Scharlemann-Schtens)

3. Stabilizations of the connected sums of Heegaard splittings.

Now let $M = V \cup_F W$ be a reducible Heegaard splitting. Then there is a 2-sphere P such that $B_1 = P \cap V$ is an essential disk in V and $B_2 = P \cap W$ is an essential disk in W. Furthermore, if P separates M into M_1 and M_2. Then B_1 separates V into V_1 and V_2, B_2 separates W into W_1 and W_2. We may assume that $V_1, W_1 \subset M_1$ and $V_2, W_2 \subset M_2$. Then $M_i \cup_P B_i^3$ has a Heegaard splitting $V_i \cup_{\partial_+ V_i} (W_i \cup_{B_i} B_i^3)$. In this case, $V \cup_F W$ is called the connected sum of the new two Heegaard splittings.

A Heegaard splitting $M = V \cup_F W$ is said to be stabilized if there are two properly embedding disks $D_1 \subset V$ and $D_2 \subset W$ such that D_1 intersects D_2 in one point; otherwise, it is said to be unstabilized. It is easy to see that $V \cup W$ is reducible if $V \cup W$ is stabilized.

An open problem on the connected sum of Heegaard splittings is the following:

Gordon's conjecture. The connected sum of two Heegaard splittings is stabilized if and only if one of the two Heegaard splittings is stabilized. (See Problem 3.91 in [Ki].)

This conjecture was proved recently by Qiu.

Theorem 1. The connected sum of two Heegaard splittings is stabilized if and only if one of the two Heegaard splittings is stabilized.

Remarks.

1) Now let M_i be a 3-manifold, and B_i^3 be a 3-ball in M_i. Then the manifold $(M_1 - intB_1^3) \cup_{\partial B_1^3 = \partial B_2^3} (M_2 - intB_2^3)$ is called a connected sum of M_1 and M_2. By Theorem 1, if one of M_1 and M_2 has different unstabilized Heegaard splittings, then M has different unstabilized Heegaard splittings.

2) K. Edwards have given a part result on Gordon's conjecture.

3) Let M be a compact 3-manifold, and S be an essential, separating closed surface in M. We denote by M_1, M_2 the two components of $M - S$. Now let $M_i = H_1^i \cup H_2^i$ be a Heegaard splitting of M_i such that $\partial_- H_2^1 = \partial_- H_1^2 = S$. Now M has a natural Heegaard splitting as follow:
$M = (\partial_- H_1^1 \times I) \cup \{1 - handles\ in\ H_1^1\ and\ H_1^2\} \cup \{2 - handles\ in\ H_2^1\ and\ H_2^2\} \cup \{3 - handles\}$.

In the other word, let H_1 be the manifold obtained by attaching all 1-handels in H_1^1 and H_1^2 to $\partial_- H_1^1 \times I$, then H_1 is a compression body with $\partial_- H_1 = \partial_- H_1^1$. Let H_2 be the manifold obtained by attaching all 2-handles in H_2^1 and H_2^2 to $\partial_+ H_1 \times I$ then capping off the possible spherical boundary components with 3-handles. Then $M = H_1 \cup H_2$ is a Heegaard splitting of M. Now we say $H_1 \cup H_2$ is the amalgamation of $H_1^1 \cup H_2^1$ and $H_1^2 \cup H_2^2$. Thus a natural question arises:

Question. Suppose that $H_1^i \cup H_2^i$ is unstabilized for $i = 1, 2$. How many times can $H_1 \cup H_2$ be stabilized simultaneously?

Some examples have been given to show $H_1 \cup H_2$ can be stabilized $g(S)$ times simultaneously. (see [KQWY].)

Generalized Gordon's Conjecture. $H_1 \cup H_2$ can be stabilized at most $g(S)$ times simultaneously.

This conjecture is still open.

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