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COMPLEXITY OF HEEGAARD SPLITTINGS

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1. Background

Let M denote a compact orientable 3-manifold and $(C_1, C_2; F)$ a genus $g \geq 2$ Heegaard splitting of M. In the 1960s, Haken [Ha68] introduced a condition of Heegaard splittings which is now said to be *reducible*. Here, $(C_1, C_2; F)$ is said to be *reducible* if there are essential disks $D_i \subset C_i$ $(i = 1, 2)$ with $\partial D_1 = \partial D_2$. Otherwise, $(C_1, C_2; F)$ is said to be *irreducible*. It is proved that if M is reducible, then any Heegaard splitting of M is reducible. The concept of weak reducibility was introduced by Casson and Gordon [CG87]. Here, $(C_1, C_2; F)$ is said to be *weakly reducible* if there are essential disks $D_i \subset C_i$ $(i = 1, 2)$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $(C_1, C_2; F)$ is said to be *strongly irreducible*. They proved in [CG87] that if a Heegaard splitting of M is weakly reducible, then either the splitting is reducible or M contains an orientable incompressible surface. In this direction, Thompson [Th99] introduced a condition called the *disjoint curve property*. Here, $(C_1, C_2; F)$ admits the *disjoint curve property* if there are essential disks $D_i \subset C_i$ $(i = 1, 2)$ and an essential loop $z \subset F$ with $(\partial D_1 \cup \partial D_2) \cap z = \emptyset$. In [Th99], she studied genus 2 closed orientable manifolds with Heegaard splittings satisfying the disjoint curve property. Moreover, Hempel [He01] introduced complexity of genus $g \geq 2$ Heegaard splittings of closed orientable 3-manifolds. It is called the 'distance' and is determined by a non-negative integer. The 'distance' is defined by using the curve complex of a Heegaard surface and is extension of the above conditions. In fact, Heegaard splittings with 'distance$=$ 0' are reducible splittings and vise versa. A Heegaard splitting has 'distance$\leq$ 1' if and only if the splitting is weakly reducible. A Heegaard splitting has 'distance$\leq$ 2' if and only if the splitting admits the disjoint curve property. He proved that if a closed orientable 3-manifold M is reducible, toroidal or Seifert fibered, then any splitting of M has 'distance$\leq$ 2'. He also showed that for any integer n, there is a closed 3-manifold with a Heegaard splittings of 'distance$>$ n'. Note that Schleimer showed in [Sc04] that for a given 3-manifold, the numbers of Heegaard splittings of 'distance$\geq$ 3' is finite.

In this paper, we consider the following condition.

Definition 1.1. A Heegaard splitting $(C_1, C_2; F)$ admits the *disjoint* (A^2, D^2)*-pair property* if there are an essential annulus A_i normally embedded in C_i and an essential disk D_j in C_j $((i, j) = (1, 2)$ or $(2, 1))$ such that ∂A_i is disjoint from ∂D_j.

We remark that the condition is essentially defined by Schleimer [Sc04].

2. Preliminaries

Throughout this paper, we work in the piecewise linear category. Let B be a submanifold of a manifold A. The notation $N(B; A)$ denotes a regular neighbourhood of B in A. The notation $|\cdot|$ denotes the number of connected components. A *surface* means a connected 2-manifold.

A simple loop/arc properly embedded in a surface is said to be *inessential* if the loop/arc cuts off a disk from the surface. A simple loop/arc properly embedded in a surface is *essential* if the loop/arc is not inessential. A disk D^2 properly embedded in a 3-manifold M is *inessential* in M if ∂D^2 is inessential in ∂M. A disk D^2 properly embedded in a 3-manifold M is *essential* in M if D^2 is not inessential in M. A 2-manifold $S(\neq D^2)$ properly embedded in a 3-manifold M is said to be *compressible in* M if there is a disk $D \subset M$ such that $D \cap S = \partial D$ and ∂D is essential in S. The disk D is called a *compression disk of* S. We say that $S(\neq D^2)$ is *incompressible in* M if S is not compressible in M. The surface $S(\neq D^2)$ is *∂-parallel in* M if $\partial M \neq \emptyset$ and S is isotoped into ∂M relative ∂S. In particular, a ∂-parallel annulus A in a 3-manifold cuts off the solid torus $A \times [0,1]$ from M. We say that $S(\neq D^2)$ is *essential in* M if S is incompressible in M and is not ∂-parallel in M. The surface $S(\neq D^2)$ is said to be *∂-compressible* in M if $\partial M \neq \emptyset$, $\partial S \neq \emptyset$ and there is a disk $\delta \subset M$ such that $\delta \cap S = \partial\delta \cap S =: \alpha$ is an essential arc in S and that $\mathrm{cl}(\partial\delta \setminus \alpha)$ is an arc in ∂M. The disk δ is called a *∂-compression disk of* S. We say that $S(\neq D^2)$ is *∂-incompressible in* M if S is not ∂-compressible in M.

A 3-manifold C is a *compression body* if there is a compact connected closed surface F such that C is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint simple loops in $F \times \{1\}$ and capping off the resulting 2-sphere boundary components by 3-handles. Then $\partial_+ C$ denotes the component of ∂C corresponding to $F \times \{0\}$, and $\partial_- C$ denotes $\partial C \setminus \partial_+ C$. If $\partial_- C = \emptyset$, then C is called a *handlebody*. We say that a surface S properly embedded in C is *normally embedded* in C if $S \cap \partial_+ C = \partial S$. It is known that a ∂-compressible essential surface normally embedded in a compression body is a disk.

We say that $(C_1, C_2; F)$ is a *Heegaard splitting* of a 3-manifold M if each of C_i $(i = 1, 2)$ is a compression body, $M = C_1 \cup C_2$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = F$. The surface F is called a *Heegaard surface* of M and the genus of F is called the genus of the Heegaard splitting. We say that $(C_1, C_2; F)$ is *stabilized* if there are essential disks D_i $(i = 1, 2)$ in C_i with $|\partial D_1 \cap \partial D_2| = 1$. It is well-known that if a genus $g \geq 2$ Heegaard splitting is stabilized, then the splitting is reducible.

If one follows the point of view in Definition 1.1, the irreducibility of Heegaard splittings can be also called the *joined (D^2, D^2)-pair property* and the weak reducibility can be also called the *disjoint (D^2, D^2)-pair property.*

3. Results

Lemma 3.1. *Let $(C_1, C_2; F)$ be a genus $g \geq 2$ Heegaard splitting of a compact orientable 3-manifold.*

1. *If $(C_1, C_2; F)$ admits the disjoint (A^2, D^2)-pair property, then the splitting admits the disjoint curve property.*
2. *If $(C_1, C_2; F)$ is weakly reducible, then the splitting admits the disjoint (A^2, D^2)-pair property.*

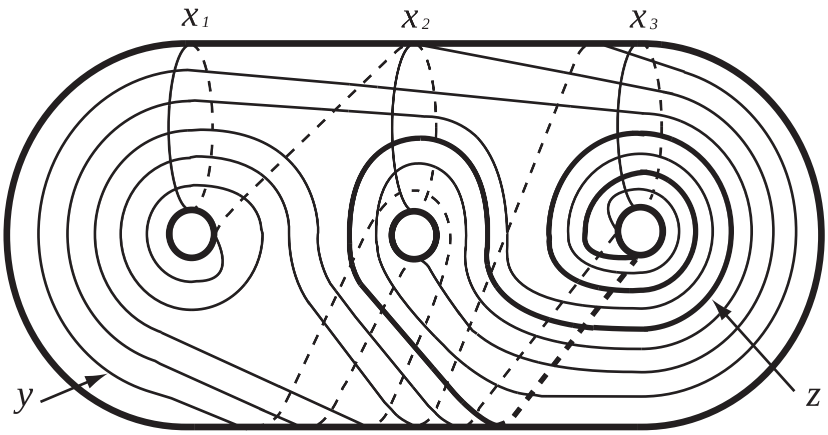

FIGURE 1

On the other hand, there is a Heegaard splitting which admits the disjoint curve property and which does not admit the disjoint (A^2, D^2)-pair property. In fact, we can give such an example. For example, Figure 1 gives an example.

Let F be a genus three closed surface illustrated in Figure 1 and x_1, x_2, x_3, y and z be loops in F as in Figure 1. Set $x_i' = \tau_y^n(x_i)$ $(i = 1, 2, 3)$, where τ_y is a Dehn twist along y. We attach 2-handles to $F \times [0, 1]$ along $x_i \times \{0\}$ and $x_i' \times \{1\}$ $(i = 1, 2, 3)$. Moreover, by capping off 3-balls along the boundary, we obtain a closed 3-manifold M. Note that $F \times \{1/2\}$ decomposes M into two handlebodies C_1 and C_2. Hence $(C_1, C_2; F)$ is a genus three Heegaard splitting of M. Note that $x_1 \cup x_1'$ is disjoint from z. It follows that $(C_1, C_2; F)$ satisfies the disjoint curve property.

We remark that *the strong rectangle condition* is very useful to prove that the splitting does not admit the disjoint (A^2, D^2)-pair property. For the definition of the strong rectangle condition, see [Ko88]. If n is sufficiently large, then we see that $\{x_1, x_2, x_3\}$ and $\{x_1', x_2', x_3'\}$ give the strong rectangle condition (*cf.* Section 7 of [Ko88]). Note that we can also prove that if a Heegaard splitting satisfies the strong rectangle condition, then the splitting does not admit the disjoint (A^2, D^2)-pair property. Hence we obtain the desired example.

Therefore we can divide the collection of strongly irreducible Heegaard splittings admitting the disjoint curve property into two sub-collections by using the disjoint (A^2, D^2)-pair property. Recall that Hempel proved that if M is reducible, toroidal or Seifert fibered, then any genus $g \geq 2$ Heegaard splitting admits the disjoint curve property. The following is our main result.

Theorem 3.2. *Let M be a compact orientable 3-manifold. If M is reducible, toroidal or Seifert fibered, then any genus $g \geq 2$ Heegaard splitting of M admits the disjoint (A^2, D^2)-pair property.*

This implies that we can slightly improve estimation for complexity of Heegaard splittings of reducible, toroidal or Seifert fibered manifolds.

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