

REDUCING AND ANNULAR DEHN FILLINGS

SANGYOP LEE

1. INTRODUCTION

Consider surfaces of non-negative Euler characteristic, i.e., sphere, disk, torus or annulus. We say that such a surface is *small* [4], and that a compact, connected, orientable 3-manifold is *simple* if it contains no essential small surfaces [11].

Let M be a 3-manifold with a torus boundary component $\partial_0 M$ and ρ a slope (the isotopy class of an essential circle) on $\partial_0 M$. We define $M(\rho)$ to be the manifold obtained by ρ -Dehn filling, that is, attaching a solid torus to M along $\partial_0 M$ so that ρ bounds a disk in the solid torus. This operation has become one of the fundamental method of constructing 3-manifolds.

Ever since Thurston [10] announced that if M is hyperbolic, then all Dehn fillings on M but finitely many yield hyperbolic manifolds, a great deal of attention has been directed towards obtaining a more precise quantification of this statement. As a part of this project, considerable energy has been devoted to finding how many Dehn filling slopes on a simple manifold yield non-simple ones. More precisely, given a simple manifold M , if π and γ are two slopes on $\partial_0 M$ such that both $M(\pi)$ and $M(\gamma)$ fail to be simple, then there are ten possible pairs of essential small surfaces that arise, and the best possible upper bounds on the distance $\Delta(\pi, \gamma)$ (the geometric intersection number of π and γ) in various cases have been established. See [4] for more detail.

Furthermore, in some choices of the pair of the involved small surfaces, there are results examining the examples of 3-manifolds which occur at or near the maximal distance. For example, Gordon [3] showed that if both $M(\pi)$ and $M(\gamma)$ contain essential tori, then $\Delta(\pi, \gamma) \leq 5$ except for four manifolds realizing $\Delta = 6, 7, 8$, and Gordon and Wu [6, 7] showed that if $M(\pi)$ contains an essential annulus and $M(\gamma)$ contains an essential torus (or annulus), then $\Delta(\pi, \gamma) \leq 3$ except for three manifolds realizing $\Delta = 4, 5$.

In this paper, we are interested in the case that $M(\pi)$ and $M(\gamma)$ contain an essential sphere and an essential annulus respectively. In this case, Wu [11] and independently Qiu [9] proved that $\Delta(\pi, \gamma) \leq 2$. Moreover, Wu also showed that if we put an additional homological condition $H_2(M, \partial M - \partial_0 M) \neq 0$, then $\Delta(\pi, \gamma) \leq 1$. In particular, this condition holds if M either has a boundary component with genus at least 2, or if it has more than two boundary tori. Thus when $\Delta(\pi, \gamma) = 2$, ∂M must be a union of two tori. On the other hand, Eudave-Muñoz and Wu [2, Theorem 2.6] constructed infinitely many examples of simple manifolds admitting reducing and annular Dehn fillings at distance 2.

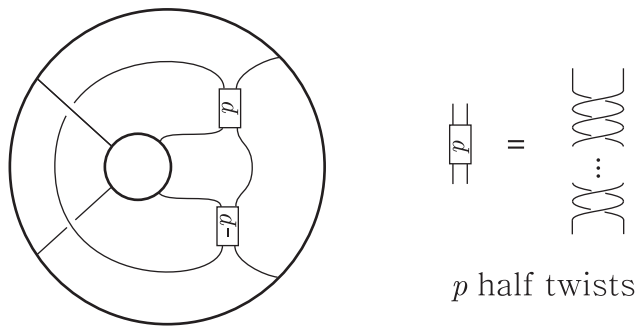


FIGURE 1

However, their examples are not all simple manifolds with such pair of Dehn fillings . In the present paper, we shall give a complete list of such manifolds.

Theorem 1.1. *If M is a simple manifold such that $M(\pi)$ is reducible and $M(\gamma)$ is annular with $\Delta(\pi, \gamma) = 2$, then M is one of the manifolds appeared in Theorem 2.1.*

2. DEHN FILLINGS CREATING ESSENTIAL SPHERES AND ANNULI

In this section, we slightly modify Eudave-Muñoz and Wu’s method in [2, Section 2] to give a list of simple manifolds admitting reducing and annular Dehn fillings at distance 2.

Following [2, Figure 2.1], let $Y = S^2 \times I$ and ξ_p the tangle in Y as shown in Figure 1. We remark that Eudave-Muñoz and Wu considered only the case that p is even. Let M_p be the double cover of Y branched along the tangle ξ_p . Then M_p is a compact 3-manifold with ∂M_p a union of two tori. For $r = \infty, -1/2$, let $\xi_p(r)$ be the tangles in [2, Figure 2.2(a),(d)] with $2p$ replaced by p , and $M_p(r)$ the corresponding double branched covering. Denote by $C(r, s)$ the cable space of type (r, s) and by $Q(r, s)$ the Seifert fibered space with orbifold a disk with two cone points of indices r and s . Then $M_p(\infty)$ is a reducible manifold, the connected sum of a solid torus and the real projective space, and $M_p(-1/2)$ is an annular manifold, the union of $C(2, 1)$ and $Q(p, -p)$ along their boundary tori.

Theorem 2.1. *The manifolds $M_p, p \geq 3$, are mutually distinct simple manifolds, each having two slopes π and γ at distance 2 such that $M_p(\pi)$ is a reducible manifold, the connected sum of a solid torus and the real projective space, and $M_p(\gamma)$ is an annular manifold, the union of $C(2, 1)$ and $Q(p, -p)$ along their boundary tori.*

3. THE GRAPHS OF INTERSECTION

From now on, we shall assume that M is a simple 3-manifold whose boundary consists of two tori, $\partial_0 M$ and $\partial_1 M$. Let π and γ be two slopes on $\partial_0 M$ such that $M(\pi)$ is reducible and $M(\gamma)$ is annular with $\Delta(\pi, \gamma) = 2$.

Let \hat{P} be a reducing sphere in $M(\pi)$ which intersects the attached solid torus V_π in a family of meridian disks. We assume that \hat{P} is chosen so that $p = |\hat{P} \cap V_\pi|$ is minimal over all such reducing spheres in $M(\pi)$. Then $P = \hat{P} \cap M$ is an essential planar surface in M ,

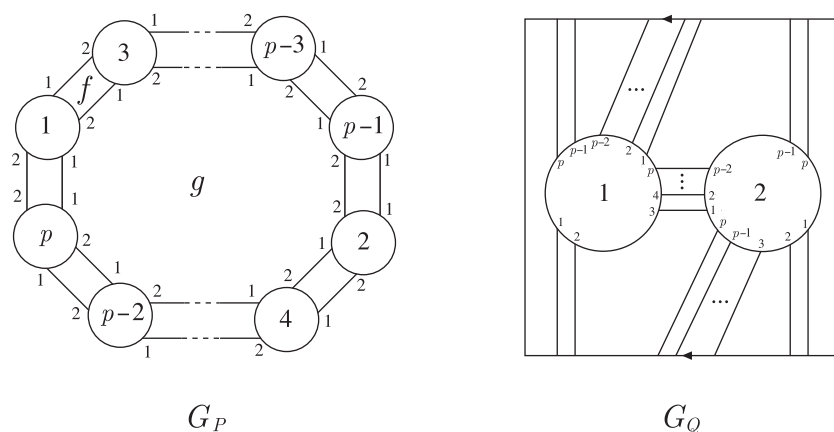


FIGURE 2

with p boundary components, each having slope π on $\partial_0 M$. Since M is simple, we have $p \geq 3$.

Similarly, let \widehat{Q} be an essential annulus in $M(\gamma)$ which intersects the attached solid torus V_γ in a family of meridian disks so that $Q = \widehat{Q} \cap M$ has a minimal number of boundary components over all such surfaces. Let $q = |\widehat{Q} \cap V_\gamma|$. Each component of $\partial Q \cap \partial_0 M$ has slope γ . Then one of the results in [8] is the following.

Theorem 3.1. $q = 2$.

By a standard argument we may assume that P meets Q transversely, in properly embedded arcs and circles such that no arc component of $P \cap Q$ is boundary parallel in either P or Q , and no circle component bounds a disk in either P or Q . We also assume that each component of ∂P meets each component of $\partial Q \cap \partial_0 M$ in $\Delta(\pi, \gamma) = 2$ points.

Let G_P be the graph in \widehat{P} with the disks of $\widehat{P} \cap V_\pi$ as (fat) vertices, and the arc components of $P \cap Q$ as edges. Similarly for G_Q . Theorem 3.1 implies that G_Q has exactly two vertices.

Let u_1, \dots, u_p (resp. v_1, v_2) be the disks of $\widehat{P} \cap V_\pi$ (resp. $\widehat{Q} \cap V_\gamma$), labelled successively when travelling along the attached solid torus. This gives a numbering of the vertices of G_P (resp. G_Q). Furthermore, it induces a labelling of the edge-endpoints in G_P and G_Q in such a way that if an edge-endpoint x in G_P or G_Q is the intersection of ∂u_i and ∂v_j , then x is labelled j in G_P , while it is labelled i in G_Q . The labels of the edge-endpoints in G_P (resp. G_Q) are considered as integers modulo q (resp. p). Then, for example, the labels in G_Q , when going around ∂v_j , appear as $1, 2, \dots, p$ repeated Δ times.

An intricate analysis of the pair of labelled graphs G_P and G_Q shows that the only possible configuration of the pair is given as shown in Figure 2.

Proof of Theorem 1.1. G_P has $p-2$ bigons bounded by positive edges, two bigons bounded by negative edges and two p -gons. Let f be a bigon and g be a p -gon as indicated in Figure 2. Using the bigon f , one can show that \widehat{Q} is separating in $M(\gamma)$. See [11, Lemma 5.4(1)]. Let X and Y be the closures of the components of $M(\gamma) \setminus \widehat{Q}$. For $Z \in \{X, Y\}$,

let $A_Z = \partial_0 M \cap Z$, $B_Z = \partial_1 M \cap Z$ and $H_Z = V_\gamma \cap Z$. Then A_Z and B_Z are annuli. One can show that ∂f and ∂g are non-separating circles in surfaces $Q \cup A_X$ and $Q \cup A_Y$, respectively.

Let N be a regular neighborhood of $\partial_0 M \cup Q \cup f \cup g$ in M . Surgering $Q \cup A_X$ along f produces an annulus parallel to B_X , and similarly surgering $Q \cup A_Y$ along g produces an annulus parallel to B_Y . Thus N is homeomorphic to M . This means that M is uniquely determined by the pair of graphs G_P and G_Q , and hence by p .

Now consider $M(\gamma)$. X is homeomorphic to a regular neighborhood of $\widehat{Q} \cup H_X \cup f$ and by [5, Lemma 3.7] it is a solid torus such that \widehat{Q} runs twice longitudinally on its boundary. On the other hand, Y is obtained from a genus two handlebody $\widehat{Q} \cup H_Y$ by attaching a 2-handle g . Calculating the fundamental group of the resulting space, one can verify that $Y = Q(p/2 + 1, -p/2 - 1)$ with $\widehat{Q} \subset \partial Y$ (see the proof of [1, Lemma 2.2.1]). Glue X and Y along \widehat{Q} to get $M(\gamma)$. Since \widehat{Q} runs twice longitudinally on the boundary of the solid torus X , the union of X and a collar of ∂Y in Y produces a cable space $C(2, 1)$. Thus $M(\gamma)$ is the union of a cable space $C(2, 1)$ and the Seifert fibered space $Q(p/2 + 1, -p/2 - 1)$ along their torus boundary. So we conclude that M is homeomorphic to $M_{p/2+1}$ in Theorem 2.1. \square

REFERENCES

- [1] J. Dean, *Small Seifert-fibered Dehn surgery on hyperbolic knots* *Algebr. Geom. Topol.* **3** (2003), 435–472
- [2] M. Eudave-Muñoz and Y.Q. Wu, *Nonhyperbolic Dehn fillings on hyperbolic 3-manifolds*, *Pacific J. Math.* **190** (1999), 261–275.
- [3] C. McA. Gordon, *Boundary slopes on punctured tori in 3-manifolds*, *Trans. Amer. Math. Soc.* **350** (1998), 1713–1790.
- [4] C. McA. Gordon, *Small surfaces and Dehn fillings*, *Geom. and Topol. Mono.* **2** (1999), 177–199.
- [5] C. McA. Gordon and J. Luecke, *Dehn surgeries on knots creating essential tori. I*, *Comm. Anal. Geom.* **3** (1995), 597–644.
- [6] C. McA. Gordon and Y.Q. Wu, *Toroidal and annular Dehn fillings*, *Proc. London. Math. Soc.* **78** (1999), 662–700.
- [7] C. McA. Gordon and Y.Q. Wu, *Annular Dehn fillings*, *Comment. Math. Helv.* **75** (2000), 430–456.
- [8] S. Lee, S. Oh and M. Teragaito, *Dehn fillings and small surfaces*, Preprint 2003. <http://de.arxiv.org/abs/math.GT/0303156>.
- [9] R. Qiu, *Reducible Dehn surgery and annular Dehn surgery*, *Pacific J. Math.* **192** (2000), 357–368.
- [10] W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, *Bull. Amer. Math. Soc.* **6**(1982), 357–381.
- [11] Y.Q. Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, *J. Diff. Geom.* **48** (1998), 407–437.

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY 207-43 CHEONGRYANGRI-DONG, DONGDAEMUN-GU SEOUL 130-722, KOREA

E-mail address: `slee@kias.re.kr`