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ON CONWAY POLYNOMIALS OF STRONGLY n-TRIVIAL KNOTS AND KNOTS OBTAINED FROM BRUNNIAN LINKS

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Abstract. Let $L = k_1 \cup \cdots \cup k_m$ be an m-component Brunnian link in S^3. Namely $L - k_i$ is an $(m-1)$-component trivial link in S^3. We denote by K_i the knot obtained from k_i by performing ± 1-twist along each component of $L - k_i$. Then it follows from the Casson surgery formula that $|a_2(K_i)|$ coincide, where a_2 denotes the second coefficient for the Conway polynomial ∇. When $m = 2$, there are no other restrictions. When $m = 3$, there are some other restrictions. When $m > 3$, $\nabla_{K_i}(z) = 1$. As an application, we show some properties of the Alexander polynomial of strongly n-trivial knots/links. We also show that any strongly n-trivial link is a boundary link and has a closed essential surface in the complement.

1. Introduction

In this note, we report recent results on strongly n-trivial knots and links (defined in §2), and review some related results in [19]. Let K be a knot in an integral homology 3-sphere H. We denote by $\Delta_K(t)$ the symmetric Alexander polynomial of K which can be written as $\Delta_K(t) = 1 + \sum a_{2i}(t^{-1/2} - t^{1/2})^{2i}$, where a_{2i} denotes the $2i$th coefficient of the Conway polynomial $\nabla_K(z)$. Then we recall that $a_2(K) = \frac{1}{2}\Delta''_K(1)$. For a rational number γ, we denote by $\chi(X; (K, \gamma))$ the object obtained from an X by γ-surgery on K.

In 1985, A. Casson introduced an integer valued invariant λ for oriented integral homology 3-spheres which satisfies the following properties:

- $\lambda(S^3) = 0$
- $\lambda(\chi(H; (K, 1/n))) - \lambda(H) = na_2(K)$.

The last equality is called the Casson surgery formula ([1], [14], [8]). Let $L_0 = k_1 \cup \cdots \cup k_m$ be an m-component Brunnian link in S^3. Namely $L_0 - k_i$ is a trivial $(m-1)$-component link. Denote by $K_i = \chi(k_i; (k_j, -1)_{j \neq i})$ the knot obtained from k_i by adding a single full twist along each component of $L_0 - k_i$. It follows from the Casson surgery formula that $a_2(K_i) = a_2(K_j)$ for every i, j because the -1-surgeries on K_i yield the same integral homology sphere. Motivated by the Casson surgery formula, the author and H. Yamada [19] showed the following:

Proposition 1.1 (cf. [19, Theorem 1.1]). *Given two polynomials*

- $f_1(z) = 1 + az^2 + b_{1,4}z^4 + \cdots + b_{1,2m_1}z^{2m_1}$
- $f_2(z) = 1 + az^2 + b_{2,4}z^4 + \cdots + b_{2,2m_2}z^{2m_2}$

where a, $b_{i,j} \in \mathbb{Z}$, there is a 2-component Brunnian link $L_0 = k_1 \cup k_2$ such that

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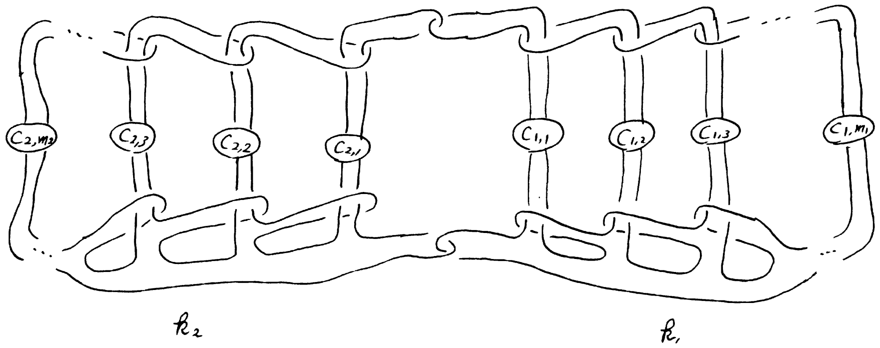
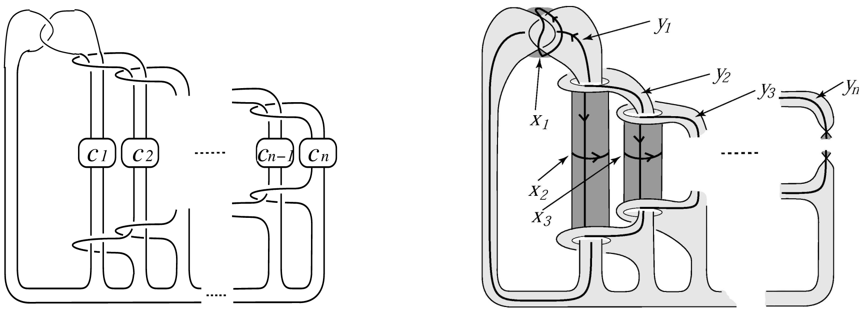


FIGURE 1. $L_0 = k_1 \cup k_2$.



(1): $K_{\vec{c}}$. (2): A basis corresponding to the matrix $A_{\vec{c}}$.

FIGURE 2

- $\nabla_{K_1}(z) = f_1(z)$
- $\nabla_{K_2}(z) = f_2(z)$ □

This was shown in the proof of [19, Theorem 1.1]. The construction of L_0 is explicit as in Figure 1, where $c_{i,j} = (-1)^j b_{i,j}$ and $\boxed{\text{c}}$ represents c full twists of the two strings.

There are several methods to realize a given knot-Conway polynomial by unknotting number one knots. See [11], [19] for example. For an n-tuple $\vec{c} = (c_1, \cdots, c_n)$, put

$$A_{\vec{c}} = \begin{pmatrix} -1 & -1 & & & & & & & \\ 0 & c_1 & 1 & & & & & & \\ & 1 & 0 & -1 & & & & & \\ & & 0 & c_2 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 0 & -1 & & \\ & & & & & 0 & c_{n-1} & 1 & \\ & & & & & & 1 & 0 & -1 \\ & & & & & & & 0 & c_n \end{pmatrix}.$$

Then by direct calculation, we have:

Lemma 1.2 (cf. [19, Lemma 3.1])**.** $\det\left(t^{1/2}A_{\vec{c}} - t^{-1/2}A_{\vec{c}}^{\mathrm{T}}\right) = 1 + \sum_{i=1}^{n}(-1)^i c_i z^{2i}$. □

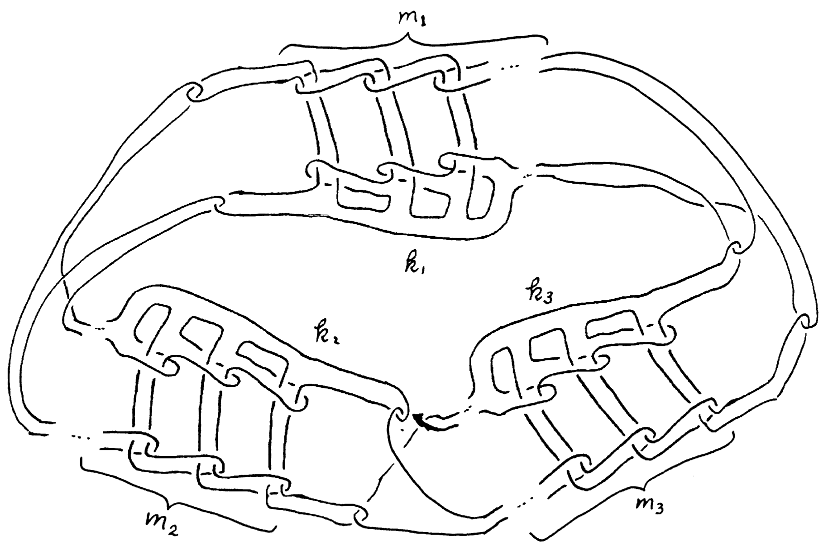


FIGURE 3. $L_0 = k_1 \cup k_2 \cup k_3$.

This calculation is not difficult. See [19, Lemma 3.1] for the details. We illustrate in Figure 2-(1) a special case of [19, Fig. 2]. Notice that the knot $K_{\vec{c}}$ has a Seifert matrix isomorphic to $A_{\vec{c}}$. Using an argument similar to that of [19], we can show that $\nabla_{K_1}(z) = 1 \pm z^{2m_1}$, $\nabla_{K_2}(z) = 1 \pm z^{2m_2}$, $\nabla_{K_3}(z) = 1 \pm z^{2m_3}$ for the 3-component Brunnian link $L_0 = k_1 \cup k_2 \cup k_3$ illustrated in Figure 3 by considering Seifert surfaces.

On the other hand, as an application of "surgical view" of the Alexander polynomial [13], we have the following well-known proposition for $m(\geq 4)$-component Brunnian links.

Proposition 1.3. *Let $L = k_1 \cup \cdots \cup k_m$ be an m-component Brunnian link. If $m \geq 4$, then $\Delta_{K_i}(t) = 1$.* □

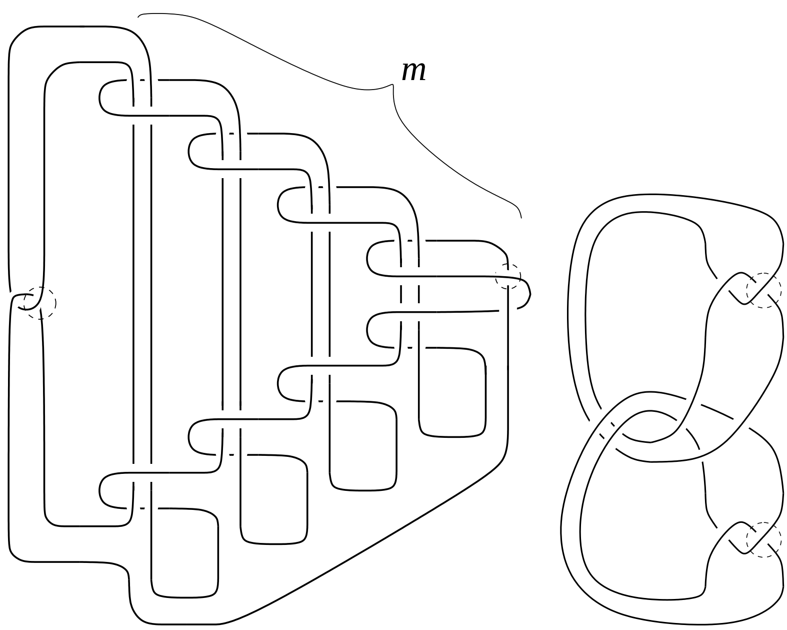
These phenomena give some algebraic restriction to strongly n-trivial knots and links.

2. On strongly n-trivial knots and links

A knot (resp. link) L is said to be *strongly n-trivial* [9, Definition 1.1] if there exists a diagram of L such that one can choose $n+1$ crossing points with the property that making the crossing changes on any $m \leq n+1$ points yields a trivial knot (resp. link). We illustrate examples of strongly 1-trivial knots and links in Figure 4.

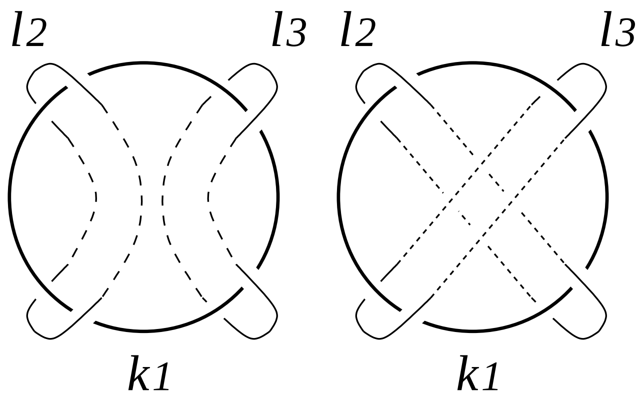
H. Howards and J. Luecke [9] constructed strongly n-trivial knots for any given natural number n by using "finger moves" [9, Section 6]. Conversely, N. Askitas and E. Kalfagianni [2] showed that any strongly n-trivial knot is obtained from the unknot by doing finger moves. Similar arguments together with several results on Dehn surgery along knots in solid tori ([6], [15], [5], [3]) work for strongly n-trivial links. (See [20] for the details.) In other words, we have:

Lemma 2.1 ([20]). *For a strongly n-trivial m-component link $L = K_1 \cup \cdots \cup K_m$, there is an $(n+m+1)$-component Brunnian link $L_0 = k_1 \cup \cdots \cup k_m \cup \ell_{m+1} \cup \cdots \cup \ell_{m+n+1}$ such that $L = \chi(k_1 \cup \cdots \cup k_m; (\ell_i, \varepsilon_i)_{m+1 \leq i \leq m+n+1})$ for some $\varepsilon_i \in \{\pm 1\}$.* □



(1): K (2): L

FIGURE 4. A strongly 1-trivial knot K with $\nabla_K(z) = 1 \pm z^{2m}$ and a strongly 1-trivial link L. When $m = 1$, K is the trefoil or the figure-eight knot.



(1): unlinked ($a_2 = 0$) (2): linked ($a_2 = \pm 1$)

FIGURE 5

It is known that any Vassiliev invariant of order $\leq n$ vanishes for strongly n-trivial knots. In [2], Askitas and Kalfagianni studied Seifert surfaces of strongly n-trivial knots by using techniques in 3-dimensional topology. They showed that if K is strongly n-trivial for $n \geq 2$, then $\nabla_K(z) = 1$ [2, Theorem 1.2]. This can be also shown by using Lemma 2.1 and Proposition 1.3. Furthermore we have that if K is strongly 1-trivial, then $a_2(K) = 0$ or ± 1 according to whether the trivializers are unlinked or linked. (cf. Figure 5) Torisu informed the author that Stanford observed a similar result. See also [10].

In [16], [17], Torisu showed the followings. See his draft [18] in this volume.

Theorem 2.2 ([16]). *If K is a strongly n-trivial 2-bridge knot, then K is the trefoil or the figure-eight knot.* □

Theorem 2.3 ([17]). *There is no strongly n-trivial 2-bridge link.* □

Now we find some other geometric restriction to strongly n-trivial links. A link is called a *boundary link* if the components bound mutually disjoint Seifert surfaces. If L is a boundary link, then $\Delta_L(t) = 0$, and the exterior $E(L)$ contains a closed incompressible surface which is not ∂-parallel. Torisu announced Theorem 2.3 in a workshop "Tohoku Musubime Seminar" held at Yamagata University in January 2004. After his talk the author showed the following:

Theorem 2.4 ([20]). *Each strongly n-trivial link is a boundary link.* □

Recall that a 2-bridge knot/link $L = S(a, b)$ in the Schubert form, $\Delta_L(-1) = a$ since the double branched covering space is the Lens space of type (a, b). Recall also that the exterior of a 2-bridge link does not contain closed essential surfaces (cf. [7], [4]). We can prove Theorem 2.3 by using any one of these facts.

Corollary 2.5. *If L is a strongly n-trivial link, then $\Delta_L(t) = 0$, the bridge index of L is greater than two, and the exterior of L contains closed essential surfaces.* □

One can easily find that the link in Figure 4-(2) is a boundary link and that there is an essential torus in the complement.

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References

[1] S. Akbulut and J. McCarthy, *Casson's invariant for oriented homology 3-spheres: an exposition*, Princeton University Press, Princeton 1990.
[2] N. Askitas and E. Kalfagianni, *On knot adjacency*, Topology Appl. **126** (2002), 63–81.
[3] J. Berge, *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$*, Topology Appl. **38** (1991), 1–19.
[4] W. Floyd and A. Hatcher, *The space of incompressible surfaces in a 2-bridge link complement*, Trans. Amer. Math. Soc. **305** (1988), 575–599.
[5] D. Gabai, *Surgery on knots in solid tori*, Topology **28** (1989), 1–6.
[6] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.
[7] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.
[8] J. Hoste, *A formula for Casson's invariant*, Trans. Amer. Math. Soc., **297** (1986), 547–562.
[9] H. Howards and J. Luecke, *Strongly n-trivial knots*, Bull. London Math. Soc., **34** (2002), 431–437.
[10] E. Kalfagianni and X.-S. Lin, *Knot adjacency and satellites*, Topology Appl., **138** (2004) 207–217.
[11] H. Murakami, *Delta-unknotting number and the Conway polynomial*, Kobe J. Math. **10** (1993), 17–22.
[12] D. Rolfsen, *Knots and links*, Publish or Perish, Inc, 1976
[13] D. Rolfsen, *A surgical view of Alexander's polynomial*, in Geometric topology (Proc. Conf., Park City, Utah, 1974), pp. 415–423. Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975.

[14] N. Saveliev, *Lectures on the topology of* 3*-manifolds: An introduction to the Casson invariant*, de Gruyter Textbook, Walter de Gruyter 1999.
[15] M. Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Diff. Geom. 29. 557-614 (1989).
[16] I. Torisu, *On strongly n-trivial* 2*-bridge knots*, To appear in Math. Proc. Camb. Phil. Soc.
[17] I. Torisu, *A remark on two-bridge links with strong triviality*, preprint.
[18] I. Torisu, *Strongly n-trivial* 2*-bridge knots and links*, in this volume.
[19] Y. Tsutsumi and H. Yamada, *Variation of the Alexander-Conway polynomial under Dehn surgery*, Topology, to appear.
[20] Y. Tsutsumi, *Strongly n-trivial links are boundary links*, preprint.

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