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ON THE COLORED JONES POLYNOMIAL OF LINKS

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Abstract. We give a formula for the N-colored Jones polynomial of an example of a nontrivial 2-component link whose (2-colored) Jones polynomial is equal to that of 2-component trivial link. We compute the N-colored Jones polynomial by using *Mathematica*.

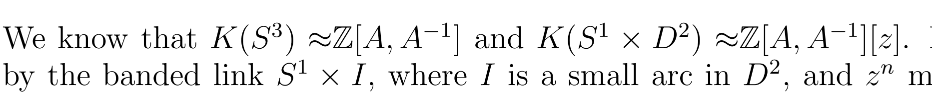
1. Introduction

The N-colored Jones polynomial of knots is a quantum invariant which can be expressed in terms of skein theory using the Kauffman bracket skein module. Recently, K. Habiro and T. Q. T. Le showed formulas for the trefoil and the figure eight knot and G. Masbaum generalized them for twist knots. Now it is important to find a formula for the colored Jones polynomial because it contribute to investigate the *Kashaev-Murakami-Murakami volume conjecture* [2][4]. In [7], We gave a formula for doubled knots, which is defined as certain satellite knot for any given knot K. Moreover, we expressed the formula by using the colored Jones polynomial of K. In this paper, we will give a formula for an example of a link, whose Jones polynomial is equal to that of 2-component trivial link by using same technique.

2. Kauffman bracket skein module

We use the notation of G. Masbaum [6]. Let A be an indeterminate and let $Z[A, A^{-1}]$ be Laurent polynomial ring. We put $a = A^2$, $\{n\} = a^n - a^{-n}$ and $[n] = \{n\}/\{1\}$. Set $[n]! = [1][2]\dots[n]$, $\{n\}! = [n]!\{1\}^n$ and $\begin{bmatrix} n \\ i \end{bmatrix} = \dfrac{[n]!}{[i]![n-i]!}$.

The *Kauffman bracket skein module* $K(M)$ of an oriented 3-manifold M is the quotient of the free $\mathbb{Z}[A, A^{-1}]$-module generated by ambient isotopy classes of banded links in M, by the following Kauffman relations:



We know that $K(S^3) \approx \mathbb{Z}[A, A^{-1}]$ and $K(S^1 \times D^2) \approx \mathbb{Z}[A, A^{-1}][z]$. Here z is given by the banded link $S^1 \times I$, where I is a small arc in D^2, and z^n means n parallel copies of z. We put $\mathcal{B} = K(S^1 \times D^2)$. There is a basis $\{\mathbf{e}_i\}_{i \geq 0}$ for $\mathcal{B}$ which is defined recursively by $\mathbf{e}_0 = 1, \mathbf{e}_1 = z, \mathbf{e}_i = z\mathbf{e}_{i-1} - \mathbf{e}_{i-2}$. Let $t : \mathcal{B} \to \mathcal{B}$ denote the *twist map* induced by a full right handed twist for solid torus. It is well known that $t(\mathbf{e}_i) = \mu_i \mathbf{e}_i$, where $\mu_i = (-1)^i A^{i^2+2i}$. Given a k-component link diagram D, and $a_1, \dots, a_k \in \mathcal{B}$. Let $< a_1, \dots, a_k >_D$ denote the *Kauffman bracket* of the linear combination of link diagrams obtained from D by replacing the ith component by a_i. Then $<, \dots, >_D$ is a k-linear form on $\mathcal{B}$.

3. The colored Jones polynomial of links

Let L be a k-component link in S^3 and D be a diagram of L. We define the *N-colored Jones polynomial of L* as follows.

$J_K(N) = (-1)^{N-1}\mu_{N-1}^{-w(D)} < \mathbf{e}_{N-1}, \dots, \mathbf{e}_{N-1} >_L$,
where $w(D)$ is the writhe of D. Put $J'_K(N) = J_K(N)/J_\circ(N)$.

Examples of the non-trivial 2-component links whose Jones polynomial is equal to that of 2-component trivial link were given by M. Thistlethwaite [5]. One of them is given in Figure 1.

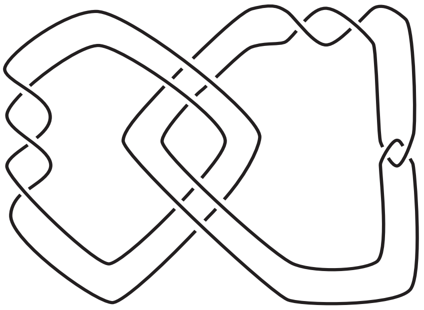


Figure 1

We obtain the following theorem.

Theorem. *Let L be a link in Figure 1. Then the colored Jones polynomial of L is given by*

$$J_L(N) = \sum_{l=0}^{N-1}(-1)^l A^{6(N^2-lN-1)} \sum_{n=0}^{N-1} \frac{A^{-n(n+3)}[N+n]!\{n\}!}{[N-1-n]!} \cdot \sum_{k=0}^{n} \frac{(-1)^k A^{4k(k+1)}[2(2lk+l+k)+1]}{[n+k+1]![n-k]!}.$$

Remark. I will show that this polynomial differs from that of a trivial link using Mathematica in Section 6.

4. The results of Habiro

K. Habiro defined a basis of B as follows [3].

$R_n = \prod_{i=0}^{n-1}(z - \lambda_{2i})$, where $\lambda_i = -a^{i+1} - a^{-i-1}$.

Then we know that

$\mathbf{e}_{N-1} = \sum_{n=0}^{N-1}(-1)^{N-1-n} \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} R_n.$

Habiro defined a twisting element w as follows.

$$w=\sum_{n=0}^{\infty}(-1)^n a^{\frac{1}{2}n(n+3)}R'_n \text{ where } R'_n = R_n/[n]!.$$

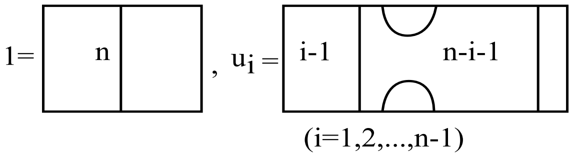
We need the following result.

Proposition [3]**.** *Let B^{even} denote the $\mathbb{Z}[A, A^{-1}]$-subalgebra of B generated by z^2. Then for every $x \in B^{even}$ $< w, x > = < t(x) >$.*

Remark. G. Masbaum generalized the formula of the element w to the formula of w^p by using skein theory [6]. By making use of the element w, we prove Theorem 1 in Section 6.

5. Graphical calculus

In this section, we explain the graphical calculus of G. Masbaum and P.Vogel to evaluate $< e_n >_K$. They used the extension of the Kauffman bracket to *admissibly colored banded trivalent graph* to evaluate the *colored link diagram* [1]. Let $\mathbb{Q}(A)$ be the field generated by the indeterminate A over the rational numbers $\mathbb{Q}$. Banded (n, n)-tangles with Kauffman relations generate a finite-dimensional associative algebra T_n over $\mathbb{Q}(A)$, which is called the *Temperley-Lieb albebra on n-strings.* T_n is generated by the following.



An integer beside an arc signifies n copies of the arc all parallel in the plane. There is a *trace map* $g : T_n \to \mathcal{B}$ given by mapping a tangle with square to the diagram in the annulus obtained by identifying the upper and lower edges of the diagram. We put $d_n = g(f^{(n)})$, where $f^{(n)}$ is the *Jones-Wenzl idenpotent* in T_n. Then it is well-known that $d_n = \mathbf{e}_n$. We regard $< \mathbf{e}_n >_K$ as the value of the diagram obtained from K by writing n beneath K and inserting one little box into K. Such a diagram is called a *colored link diagram.* The *colored banded trivalent graph* is as follows. A *color* is just an nonnegative integer. A triple of colors (a, b, c) is *admissible* if $a + b + c = 0$ (mod 2) and $a + b \geq c \geq |a - b|$. Let D be a planar diagram of a banded trivalent graph. An admissible colorings of D is an assignment of colors to the edges of D so that at each vertex, the three colors meeting there form an admmisible triple. The Kauffman bracket of D is defined to be the bracket of the expansion of D obtained as follows. The expansion of an edge colored n consists of n parallel strands with a copy of the *Jones-Wenzl idenpotent* $f^{(n)}$ inserted. The idempotent is represented by a little box and each vertex is expanded as in Figure 1.

Let (a, b, c) be admissible. The *internal colors* i, j, k are defined by $i = (b+c-a)/2$, $j = (a+c-b)/2$, $i = (a+b-c)/2$. Let A, B, C, D, E, F be colors and set $\Sigma = A+B+C+D+E+F$ and $a_1 = (A+B+E)/2, a_2 = (B+D+F)/2, a_3 = (C+D+E)/2, a_4 = (A+C+F)/2, b_1 = (\Sigma - A - D)/2, b_2 = (\Sigma - E - F)/2, b_3 = (\Sigma - B - C)/2$. G. Masbaum and P. Vogel showed the following formulas [1].

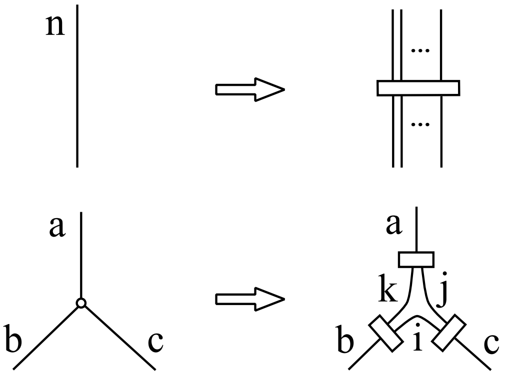
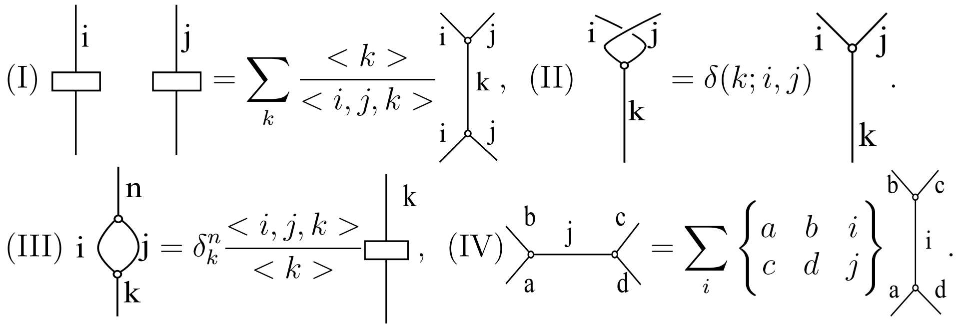


FIGURE 2



Here we set

$< k >= (-1)^k [k+1]$,

$< a, b, c >= (-1)^{i+j+k} \dfrac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}$,

$\delta(c; a, b) = (-1)^k A^{ij-k(i+j+k+2)}$.

$$\left\langle \begin{matrix} A & B & E \\ D & C & F \end{matrix} \right\rangle = \frac{\prod_{i=1}^{3} \prod_{j=1}^{4} [b_i - a_j]!}{[A]![B]![C]![D]![E]![F]!} \begin{pmatrix} a_1\ a_2\ a_3\ a_4 \\ b_1\ b_2\ b_3 \end{pmatrix},$$

$$\begin{pmatrix} a_1\ a_2\ a_3\ a_4 \\ b_1\ b_2\ b_3 \end{pmatrix} = \sum_{min(b_i) \geq \zeta \geq max(a_j)} \frac{(-1)^{\zeta} [\zeta+1]!}{\prod_{i=1}^{3} [b_i - \zeta]! \prod_{i=1}^{4} [\zeta - a_j]!},$$

and $\left\{\begin{matrix} a & b & i \\ c & d & j \end{matrix}\right\} = \dfrac{< i > \left\langle \begin{matrix} i & b & c \\ j & d & a \end{matrix} \right\rangle}{< i, a, d >< i, b, c >}$.

6. PROOF OF THEOREM

Let L be the link dscribed in Figure 1. We use the surgery description as in Figure 3.

The colored Jones polynomial is expanded about the component on the left-hand side in Figure 3 by making use of formulas in Section 5 as follows.

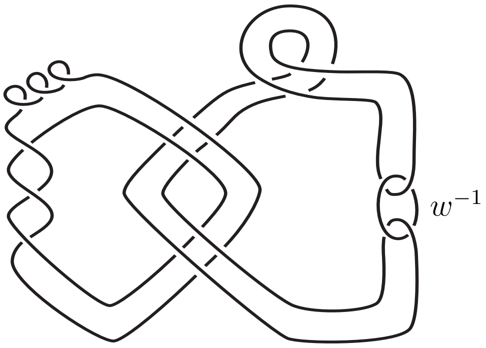
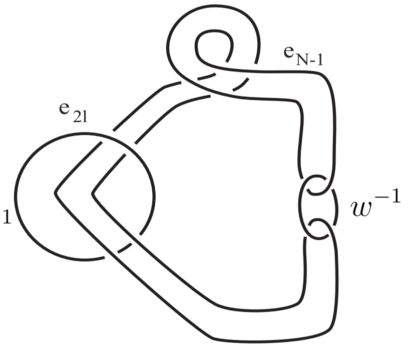


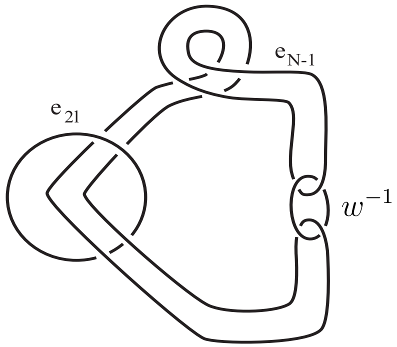
FIGURE 3

$$J_L(N) = (-1)^{N-1} < e_{N-1}, e_{N-1} >_L$$

$$= \sum_{l=0}^{N-1} \delta(2l; N-1, N-1)^{-3} \mu_{N-1}^{3}$$



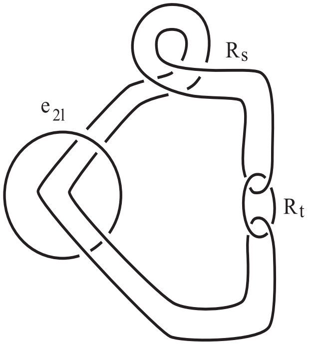
$$= \sum_{l=0}^{N-1} (-1)^l A^{6N^2-6lN-6}$$



Now we need the following lemma.

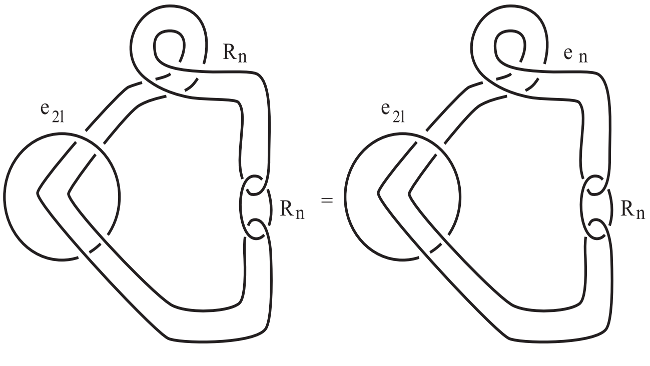
Lemma.

(1)



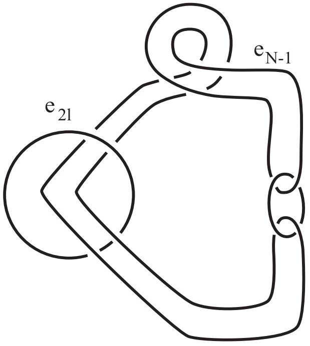
$= 0$ if $s \neq t$.

(2)



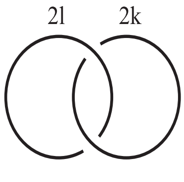
Proof. Since each component of the link bounds a disk which gone through by the other component twice, we can show the first equation using a property that circling with R_n annihilates all even polynomial in z of degree $2q$ $(q < n)$. Since the degree of $R_n - e_n$ is smaller than n, the second equation can be also shown using a property of R_n.

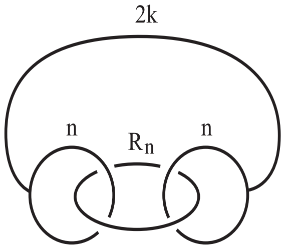
By using Lemma, we obtain that



$$w^{-1} = \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \frac{A^{-n(n+3)}}{\{n\}!}$$

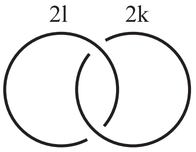
$$\cdot \sum_{k=0}^{n} \frac{< 2k >}{< n, n, 2k >} \frac{A^{4k(k+1)}}{< 2k >} \Bigl\{$$





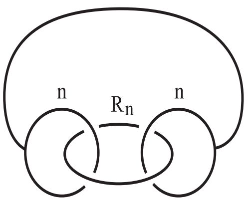
$$\Bigr\}.$$

Here by making use of the formulas in Section 5, we can show that



$$=< 4lk + 2l + 2k > \text{ and}$$

$$\frac{< 2k >}{< n, n, 2k >}$$



$$= \frac{[2k+1]([n]!)^2}{[n+k+1]![n-k]!} < e_{2n}, R_n > .$$

It is known that $< e_{2n}, R_n >= (-1)^n \frac{\{2n+1\}!}{\{1\}}$. (This equality is due to K. Habiro [3].) This completes the proof.

7. COMPUTATION

By using Mathematica, we calculated the N-colored Jones polynomial of Theorem when $N = 2, 3, 4, 5$ and 6 as follows.

$$J_L(2) = 2 + \frac{1}{A^4} + A^4$$

$$\begin{aligned} J_L(3) = {} & 2 - \frac{1}{A^{32}} + \frac{1}{A^{24}} - \frac{1}{A^{20}} + \frac{2}{A^{12}} + \frac{2}{A^8} + \frac{1}{A^4} + 2A^4 - A^6 + A^8 - A^{10} + A^{12} + A^{16} + A^{20} - A^{22} - \\ & A^{26} - A^{30} + A^{32} - A^{34} - A^{38} + A^{40} + 2A^{44} - A^{46} + A^{48} - A^{50} + A^{52} - A^{68} + A^{70} - A^{72} + A^{76} - A^{78} + A^{80} \end{aligned}$$

$$\begin{aligned} J_L(4) = {} & \frac{1}{A^{68}} - \frac{2}{A^{60}} - \frac{2}{A^{56}} + \frac{1}{A^{52}} + \frac{2}{A^{48}} - \frac{1}{A^{44}} - \frac{2}{A^{40}} + \frac{1}{A^{32}} + \frac{2}{A^{28}} + \frac{1}{A^{24}} + \frac{1}{A^{16}} + \frac{3}{A^{12}} + \\ & \frac{4}{A^8} - \frac{1}{A^4} + 2A^4 + 3A^8 + A^{12} + 2A^{20} + 2A^{24} - 2A^{28} - 3A^{32} - 2A^{36} + A^{40} + A^{44} + A^{56} + 3A^{60} + \\ & A^{64} - A^{68} - 3A^{72} + A^{76} + A^{80} + A^{84} - A^{88} + A^{96} - A^{108} - A^{124} + A^{128} + A^{136} - A^{144} - A^{152} + A^{156} \end{aligned}$$

$$\begin{aligned} J_L(5) = {} & \frac{1}{A^{108}} + \frac{1}{A^{104}} - \frac{2}{A^{96}} - \frac{2}{A^{92}} - \frac{1}{A^{66}} - \frac{1}{A^{64}} - \frac{1}{A^{60}} + \frac{1}{A^{58}} - \frac{1}{A^{56}} + \frac{2}{A^{54}} + \frac{2}{A^{50}} + \frac{2}{A^{48}} - \frac{2}{A^{46}} - \\ & \frac{3}{A^{42}} + \frac{1}{A^{40}} - \frac{2}{A^{38}} + \frac{1}{A^{36}} + \frac{2}{A^{34}} + \frac{1}{A^{32}} + \frac{4}{A^{30}} + \frac{2}{A^{28}} + \frac{1}{A^{24}} - \frac{2}{A^{22}} + \frac{1}{A^{20}} - \frac{2}{A^{18}} + \frac{2}{A^{16}} + \frac{3}{A^{12}} + \\ & \frac{1}{A^{10}} + \frac{3}{A^8} + \frac{2}{A^4} + \frac{1}{A^2} + A^4 + 3A^8 + 6A^{12} - 2A^{14} + 5A^{16} - 2A^{18} - A^{20} - 2A^{22} - A^{24} - 2A^{26} + \\ & 2A^{28} - 2A^{30} + 4A^{32} - 3A^{34} + 3A^{36} - 3A^{38} - 3A^{42} - 2A^{46} + 2A^{48} - A^{50} + 3A^{52} - A^{54} + 4A^{56} - \\ & 2A^{58} + 2A^{60} - 3A^{62} + A^{64} - 2A^{66} + 2A^{68} - A^{70} + A^{72} + 3A^{76} + 4A^{80} - 2A^{82} + 3A^{84} - 2A^{86} - \\ & 2A^{90} - 3A^{92} - A^{94} + A^{96} + A^{98} + 5A^{100} + A^{102} + 2A^{104} - A^{106} - A^{108} - 3A^{110} - 3A^{112} - 2A^{114} - \\ & A^{118} + 3A^{120} + A^{122} + 2A^{124} - A^{126} + A^{128} - 3A^{130} - A^{132} - 2A^{134} + 2A^{136} - 2A^{138} + 4A^{140} - \\ & A^{142} + 3A^{144} - A^{146} + A^{148} - 2A^{152} + 2A^{154} - 2A^{156} + A^{158} + A^{160} - A^{162} + 2A^{164} - 2A^{166} + A^{168} - \\ & A^{170} - 2A^{172} + 2A^{174} - 4A^{176} + 3A^{178} - 2A^{180} + A^{182} + A^{184} - A^{186} + 3A^{188} - 2A^{190} + 2A^{192} - \\ & A^{194} - A^{196} + A^{202} + A^{204} - A^{210} + A^{212} - A^{218} - A^{220} + A^{232} + A^{234} - A^{240} + A^{242} - A^{248} - A^{250} + A^{256} \end{aligned}$$

$$
\begin{aligned}
J_L(6)= \\
\frac{1}{A^{180}}\ & -1+A^{8}+A^{12}+A^{16}-A^{24}-2A^{28}+A^{36}+3A^{40}+2A^{44}-A^{48}-5A^{52}-6A^{56}+3A^{64}+5A^{68}+3A^{72}- \\
& 4A^{76}-5A^{80}-3A^{84}+4A^{92}+4A^{96}+A^{100}-4A^{104}-4A^{108}-3A^{112}+4A^{120}+3A^{124}+2A^{128}+ \\
& A^{132}-2A^{136}-2A^{140}+2A^{144}+5A^{148}+6A^{152}+5A^{156}+A^{160}-A^{164}+A^{168}+6A^{172}+7A^{176}+ \\
& 2A^{180}-A^{184}-A^{188}-A^{192}+2A^{196}+3A^{200}+A^{204}-2A^{208}-4A^{212}-A^{216}+3A^{224}+2A^{228}- \\
& 3A^{232}-4A^{236}-2A^{240}+4A^{244}+5A^{248}+A^{252}-3A^{260}-2A^{264}+4A^{268}+6A^{272}+4A^{276}+2A^{280}- \\
& A^{284}-5A^{288}-3A^{292}+2A^{300}+3A^{304}+A^{308}-3A^{312}-5A^{316}-3A^{320}+3A^{328}+4A^{332}+3A^{336}- \\
& 4A^{344}-4A^{348}+3A^{356}+5A^{360}+2A^{364}-3A^{368}-3A^{372}-A^{376}+A^{380}+2A^{384}-A^{392}+A^{400}+ \\
& A^{404}-A^{424}-A^{428}-A^{432}-A^{436}+A^{444}+2A^{448}+A^{452}-A^{460}-2A^{464}-A^{468}+2A^{472}+2A^{476}-A^{480}- \\
& A^{484}-A^{492}-A^{496}+A^{500}+2A^{504}-2A^{520}-A^{524}+A^{528}+A^{532}+A^{540}+A^{544}-A^{548}-2A^{552}+A^{560}
\end{aligned}
$$

REFERENCES

[1] G. Masbaum and P. Vogel, 3-valent graphs and the Kauffman bracket, Pacific J. Math. 164, (1994) 361-381.

[2] R. M. Kashaev, The hyperbolic volume of knots from quantum dilogarithm, Lett. Math. Phys, 39 (1997) 269-275.

[3] H. Habiro, On the colored Jones polynomial of some simple links, In: Recent Progress Towardsthe Volume conjecture, Research Institute for Mathematical Sciences (RIMS) Kokyuroku 1172, September (2000).

[4] H. Murakami and J. Murakami, The colored Jones polynomial and the simplicial volume of a knot, Acta Math. 186, (2001) 85-104 .

[5] M. Thistlethwaite, Links with trivial Jones polynomial, J. Knot Theory and its Ramifications 1 (1992) 407-499.

[6] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebraic and Geometric Toology. 3, (2003) 537-556.

[7] T. Tanaka, A formula for the colored Jones polynomial of links, preprint.

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