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CLASPER MOVES AMONG RIBBON 2-KNOTS CHARACTERIZING THEIR FINITE TYPE INVARIANTS

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A *1-knot* is an embedded circle into $\mathbf{R}^3$. A *ribbon 1-knot* is a 1-knot of special kind which bounds a disk called ribbon disk. A *ribbon disk* is an immersed 2-disk into $\mathbf{R}^3$ having only ribbon singularities. Further, a *2-knot* is a locally flat embedded 2-sphere into $\mathbf{R}^4$. A *ribbon 2-knot* is a 2-knot one of whose projections into $\mathbf{R}^3$ has only singularities consisting only of transversal double points. Therefore the singularity set of some projection of a ribbon 2-knot forms disjoint circles.

In 1989-90, Vassiliev [Vas90] and Goussarov [Gus91] independently developed the theory of finite type 1-knot invariants. Let A be an abelian group. We say an invariant $f : \mathbf{Z}\{\text{knots}\} \to A$ is *of type k* if $\sum_{S' \subset S}(-1)^{|S'|} f(K_{S'}) = 0$ for any 1-knot K, where S is a set of $k+1$ crossings on a diagram of K, and $K_{S'}$ is a 1-knot obtained from K by performing crossing changes on the diagram at each crossing in S'. If k is finite, then the invariant f is said to be *of finite type*. It is known that the set of all invariants of type k is generated by invariants called *additive invariants*, and that the set of all 1-knots modulo additive invariants of type k forms an abelian group generated by diagrams having one 1,3-valent tree, whose multiplication is induced by connected sum.

Habiro [Hab94, Hab97, Hab00] introduced a family of some equivalence relations among 1-knots and reconstructed the notion of finite type 1-knot invariants. Namely, he showed that the classification of 1-knots by invariants of type k equals the classification of 1-knots by the equivalence relation that two 1-knots are related by a sequence of local moves called Habiro's C_{k+1}-moves (we will say that they are *C_{k+1}-equivalent*). (Goussarov [Gus94] obtained similar result using his *Y-graphs.*) *C_k-move* is a local moves among 1-knots defined by using Habiro's clasper. As an application of this result, for example, one can easily construct many 1-knots not distinguished by any invariants of type k. In his proof, it is shown that the set of all 1-knots modulo the C_{k+1}-equivalence forms an abelian group generated by diagrams having one 1,3-valent tree called *tree clasper*, whose multiplication is induced by connected sum. Roughly spe aking, a clasper on a 1-knot is an embedded framed arc whose ends are attached to the 1-knot and causing crossing change on the 1-knot along the arc.

As an analogue of finite type knot invariants, Habiro, Kanenobu and Shima [HKS99] introduced the notion of finite ribbon type invariants of ribbon 2-knots and ribbon 1-knots. For ribbon 2-knots, it is defined similarly as usual finite type invariant by thinking of a crossing change as an operation of exchanging up and down along a crossing circle, and for ribbon 1-knots, it is defined by thinking of a crossing change as an operation of unclasping a ribbon singularity on the ribbon disk. It is shown by Habiro and Shima [HS99] that any invariant of ribbon 2-knot

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is finite ribbon type invariant if and only if it is a polynomial of the coefficients of the normalized Alexander polynomial.

In our talk we prove the ribbon version of Habiro's result. Namely, we introduce the notion of claspers for ribbon 2-knots and ribbon 1-knots, and the class of local moves called RC_k-moves, and we show the following theorem reconstructing the notion of finite ribbon type invariants.

Theorem A. *Two ribbon p-knots ($p = 1$ or 2) are not distinguished by any invariant of finite ribbon type k if and only if they are related by a sequence of RC_{k+1}-moves.*

Our clasper is defined to be, for ribbon 2-knots, about an arc connecting two circles on a ribbon 2-knot and for ribbon 1-knots, an arc connecting one point and a pair of two points on a ribbon 1-knot. We think of a clasper as an object causing a crossing change. In the definition of claspers, we color each end of a clasper different colors, for example, black and white to give each end different meaning. We define an RC_k-move as crossing changes along all claspers in a union of claspers forming a 1,3-valent tree and keeping some rules of linking and of combination of colors (we call such a union of claspers an *orientable tree clasper*). We show that the set of RC_k-equivalence classes of ribbon p-knots, where the *RC_k-equivalence* is induced by RC_k-moves, forms an abelian group generated by diagrams having precisely one orientable tree clasper, and we reconstruct the notion of finite ribbon type invariants.

One may see by definition that any 1-knot invariant of finite type k restricts to a ribbon 1-knot invariant of finite ribbon type k. Then restricted to ribbon 1-knots, whether the space of all finite ribbon type invariants is more extensive than that of all finite type invariants is a non trivial problem. Habiro and Shima conjectured in [HS99] that the former might be at most the latter, that is, the former and the latter might coincide on ribbon 1-knots. We prove this conjecture in the case of $\mathbf{Q}$-valued invariants.

Theorem B ($\mathbf{Q}$-valued version of Habiro-Shima's conjecture)**.** *Any $\mathbf{Q}$-valued ribbon 1-knot invariant of finite ribbon type k extends uniquely to a knot invariant of finite type k.*

The proof is as follows. We introduce the two spaces of 1,3-valent graphs corresponding to restrictions of finite type invariants and finite ribbon type invariants respectively, and we show the two spaces coincide. The graph space corresponding to restrictions of finite type invariants is a subspace of the graph space $\mathcal{A}/\mathrm{FI}$, where $\mathcal{A}/\mathrm{FI}$ is the space where the Kontsevich invariant [Kon93] takes values in. Then the Kontsevich invariant induces the universal finite ribbon type invariant, that is, any invariant of finite ribbon type k can be presented as a composition of the degree $\leq k$ part of the (restricted) Kontsevich invariant and some $\mathbf{Q}$-valued weight system of graphs. This is a restriction of a knot invariant of finite type k and thus completes the proof of extendability. The uniqueness of the extension follows by the result of Ng [Ng95]. We use our claspers in the proof.

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