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PLANE CURVES OF A CERTAIN TYPE WHICH GIVES BERGE'S KNOTS YIELDING LENS SPACES

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1. Introduction

In [Bg], John Berge described some families of knots in 3-sphere S^3 which have non-trivial Dehn surgeries yielding lens spaces. In [Y1], we studied its subfamily $\{k^+(a,b)\}$ (and also another family $k^-(a,b)$) from the view point of Kirby calculus on framed links, where (a,b) is a coprime pair of integers with $0 < a < b$. Each knot $k^+(a,b)$ is contained in the fiber surface F^+ of the right-handed trefoil. In the final section of [Y1], we pointed out that each knot $k^+(a,b)$ is in the class of *divide knots* defined by N. A'Campo and gave a divide (a plane curve) $P(a,b)$ giving $k^+(a,b)$ but we gave no general proof, since the main purpose of [Y1] was not divide knot theory. Here we prove that the divide $P(a,b)$ gives the knot $k^+(a,b)$ by showing a lemma "blowing-down = adding square" in a special case, in which the divide is obtained by cut out from a lattice in the plane, see also [Y1, Y2, Y3][1].

The *divide* is a relative, generic immersion of a 1-manifold in a unit disk in $\mathbf{R}^2$. N. A'Campo formulated the way to associate to each divide C a link $L(C)$ in the 3-dimensional sphere S^3 ([A1, A2, A3, A4]):

$$L(C) = \{(u,v) \in TD | u \in C,\, v \in T_uC,\, |u|^2 + |v|^2 = 1\} \subset S^3.$$

Thus divide theory is deeply related to algebraic geometry and singularity of complex curves. The class of links of divides properly contains the class of the links arising from isolated curve singularities. M. Hirasawa developed more geometric and knot theoretic method in [H].

In this paper, we draw only curves C but the disk and we say "C gives a knot K" if $L(C) = K$. Here we study plane curves of type $X \cap \mathcal{R}$, which is cut out from the 45° lattice X $(:= \{(x,y) \in \mathbf{R}^2 | \cos \pi x = \cos \pi y\})$ as the intersection with a (connected) union $\mathcal{R}$ of rectangles in the plane. Throughout this paper, we always assume that each side of any rectangles in $\mathcal{R}$ is pallarell to either x-axis or y-axis, that each corner and terminal point of sides and edges is at a lattice point and that any pair of rectangles in $\mathcal{R}$ are disjoint or share only one edge (a part of the sides). If $\mathcal{R}$ is as in the left figure in Figure 2, whose area is $a^2 + ab + b^2$, and choose the place such that the curve $X \cap \mathcal{R}$ is an immersed arc, we call the curve $P(a,b)$. We will prove the follwing:

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[1]The author's talk on 18 Feb.'04 is a short survey of [Y1, Y2, Y3].

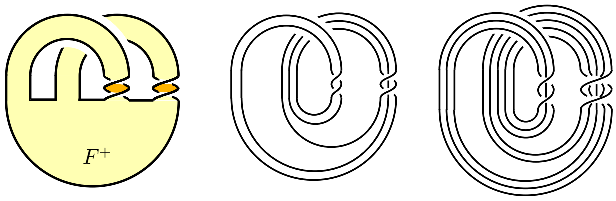

Figure 1. F^+, $k^+(2,3)$ and $k^+(3,5)$

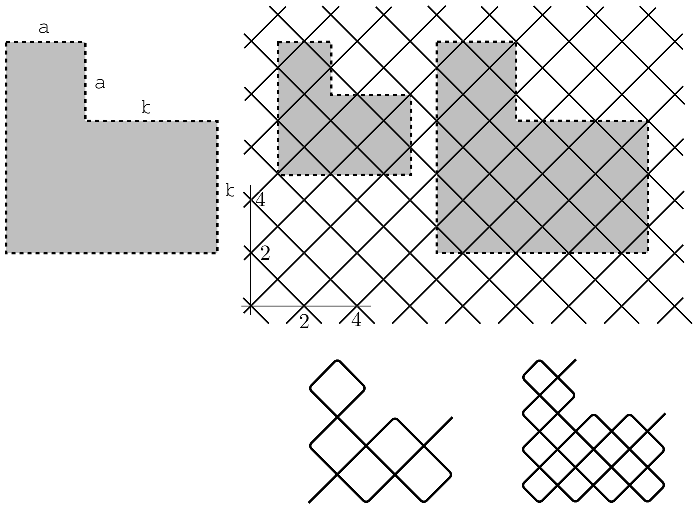

Figure 2. $\mathcal{R}$, $P(2,3)$ and $P(3,5)$

Theorem 1.1. *For a coprime pair (a, b) of positive integers with $0 < a < b$, the curve $P(a, b)$ gives the knot $k^+(a, b)$, i.e., $L(P(a, b)) = k^+(a, b)$.*

Note that $(a^2 + ab + b^2)$-surgery along $k^+(a, b)$ is a lens space (see [Bg], [Y1]).

2. Divide theory and Blow-down

We start with an operation blowing-down. Roughly, it is the following coordinate transformation:

$$(x, y) \to (x', y'); \quad \begin{cases} x' = x \\ x'y' = y \end{cases} .$$

Such transformation changes a plane curve from $f(x, y) = 0$ to $x^m f(x, y/x) = 0$, where x^m is multiplied to make it a polynommial. For example, $y^2 = x + \epsilon$ becomes (via $(y/x)^2 = x + \epsilon$) to $y^2 = x^3 + \epsilon x^2$. Topologically it changes a curve as "turn the left half of the y-axis upside down and concentrate all crossing points between the original curve and y-axis to the origin". Since in the divide knot theory, a regular isotopy of the curve with no passing though of self-tangency does not change the corresponding knots, we can purterb the resulting curve and cancel the multuiple (≥ 3) crossing without changing the corresponding knots.

On the other hand, from the view point of knot theory or framed links, blowing-down corresponds to a right-handed full twisting along the trivial circle (called

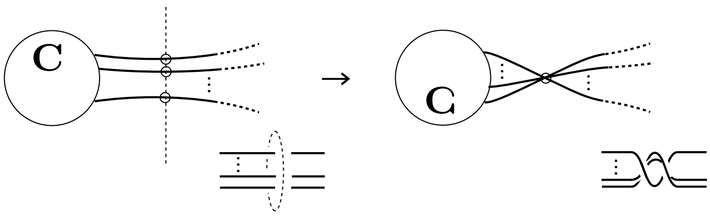

FIGURE 3. Blow-down

"exceptional curve" in the opposite operation blowing-up) that corresoponds to the axis, see the below half of Figure 3. Now, we study blowing-down in the case in which the curve is of type $X \cap \mathcal{R}$.

Before stating the next lemma, we define some notaions: Let l be an edge in $\partial\mathcal{R}$ and $\bar{l}$ its straight prolongation. From $(\bar{l}, l)$, we construct $\hat{l}$ by pushing the interior of l into $\mathcal{R}$, which part we call l_i, and pushing the compliment of l in $\bar{l}$ outward $\mathcal{R}$ both slightly, see the first figure in Figure 4.

Lemma 2.1. *Suppose that a curve $X \cap \mathcal{R}$ gives a knot K. Let l be an straight edge in $\partial\mathcal{R}$ such that $\hat{l} \cap \mathcal{R} = l_i \cap \mathcal{R}$, where $\hat{l}$ and l_i are as above. By $R(l)$ we denote the square one of whose side is l and which is on the opposite side from $\mathcal{R}$. Then $X \cap (\mathcal{R} \cup R(l))$ gives a knot obtained from K by a right-handed full twisting along the trivial circle that corresoponds to $\hat{l}$.*

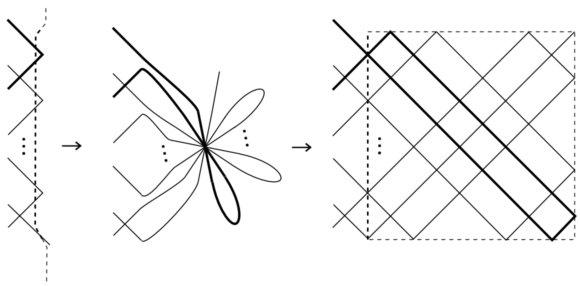

FIGURE 4. Blow-down = adding square

In [GHY] (historically in [AGV] and [GZ], or by [CP]), it is proved that if $\mathcal{R}$ is a simple rectangle of size $a \times b$, then $X \cap \mathcal{R}$ is a torus knot $T(a, b)$ of type (a, b) Thus, Theorem 1.1 is proved by blowing-down twice from a simple rectangle or $T(a, b)$, where two axes (circles) of the blowing-downs are as in Figure 5. Note that each circle (with ignoring the other!) is isotopic to the core of a solid torus bounded by the standard torus that contains $T(a, b)$.

Note that after the first blowing-down, the knot is still a torus knot, although the type is changed to $T(a + b, a)$ or $T(a + b, b)$ according to which circle is the first choice. Thus the knot $k^+(a, b)$ is obtained from a torus knot by one blowing-down. Further more, in this case, the knot is in the class of *twisted torus knots* (see [D]).

Corollary 2.2. *The knot $k^+(a, b)$ equals to a twisted torus knot $T(a + b, b; a, 1)$ and also to $T(a + b, a; b, 1)$, where $T(p, q; r, s)$ is a knot obtained from a torus knot $T(p, q)$ by s full twisting of r strings in p parallel strings of $T(p, q)$.*

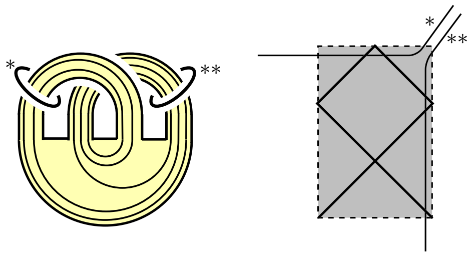

FIGURE 5. From a torus knot to a Berge's knot

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REFERENCES

[A1] N. A'Campo, *Generic immersion of curves, knots, monodromy and gordian number*, *Inst.Hautes Etudes Sci. Publ.Math.* **88** (1998) 151–169.

[A2] N. A'Campo, *Planar trees, slalom curves and hyperbolic knots*, *Inst.Hautes Etudes Sci. Publ.Math.* **88** (1998) 171–180.

[A3] N. A'Campo, *Real deformations and complex topology of plane curve singularities*, *Ann. de la Faculte des Sciences de Toulouse* **8** (1999) 5–23.

[A4] N. A'Campo, *Quadratic vanishing cycles, reduction curves and reduction of the monodromy group of plane curve singularities*, *Tohoku Math. J.* **53** (2001) 533–552.

[AGV] V.I.Arnold, S.M.Gusein-Zade and A.N.Varchenko, Singularities of Differentiable Maps, Volume II. *Monographs in Mathematics.* **83** (Birkhauser Boston, Inc., Boston, MA., 1988).

[Bg] J. Berge, *Some knots with surgeries yielding lens spaces*, (Unpublished manuscript, 1990).

[CP] O. Couture and B. Perron, *Representative braids for links associated to plane immersed curves*, *J. Knot Theory Ramifications* **9** (2000) 1–30.

[D] J. Dean, *Hyperbolic knots with small Seifert-fibered Dehn surgeries*, (Unpublished manuscript, 1996).

[GI1] W. Gibson and M. Ishikawa, Links and gordian numbers associated with generic immersions of intervals, *Topology Appl.* **123** No.3 (2002) 609-636.

[GI2] W. Gibson and M. Ishikawa, Links of oriented divides and fibrations in link exteriors, *Osaka J. Math.* **39** No.3 (2002) 681-703.

[GHY] H. Goda, M. Hirasawa and Y. Yamada, *Lissajous curves as A'Campo divides, torus knots and their fiber surfaces*, *Tokyo J. Math.* **25** No.2 (2002) 485–491.

[GZ] S. M. Gusein-Zade, *Intersection matrices for certain singularities of functions of two valuables*, *Functional Analysis and its Appl.* **8** (1974) 10–13.

[H] M. Hirasawa, *Visualization of A'Campo's fibered links and unknotting operations*, *Topology and its Appl.* **121** (2002) 287–304.

[Y1] Y. Yamada, *Berge's knots in the fiber surfaces of genus one, lens spaces and framed links*, preprint (2003).

[Y2] Y. Yamada, *Finite Dehn surgery along A'Campo's divide knots* , preprint (2004).

[Y3] Y. Yamada, *Plane slalom curves of a certain type, pretzel links and Kirby-Melvin's Grapes*, preprint (2004).

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