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A CONSTRUCTION OF CLASSIFYING SPACES FOR P-ADIC GROUP ACTIONS

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Abstract. A generalization of the Hilbert's fifth problem is the Hilbert-Smith conjecture. A natural approach is to work on the classifying space of p-adic integers. However, the well known Milnor construction of classifying space of p-adic integers is not locally connected, hence won't help to solve the conjecture. I give a new construction of classifying spaces for p-adic group actions.

1. Introduction

Group actions on manifolds have been extensively studied . For instance, Hilbert's fifth problem asks whether every (finite dimensional) locally Euclidean topological group is necessarily a Lie group. It was solved affirmatively by Gleason. Montgomery and Zippin provided the solution for locally compact groups in 1952.

Theorem: ([7]) A locally compact, finite dimensional, locally-connected group is a Lie group.

One major open problem in geometric topology is a generalization of the Hilbert's fifth problem, called the Hilbert-Smith conjecture:

Hilbert-Smith Conjecture: No p-adic group can act effectively on a manifold. Equivalently, no compact manifold M admits a self homeomorphism h such that:
(1) each orbit $\{h^n(x)\}$ has small diameter in M and,
(2) $\{h^n \mid n \in \mathbb{Z}\}$ is a relatively compact subgroup of the group of all homeomorphism $M \to M$.

It follows from the work of Newman and Smith that it is sufficient to prove the special case when the topological group is a p-adic integer group $\widehat{\mathbb{Z}}_p = \underleftarrow{lim}\{\phi_n\colon\ \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}\}$, where ϕ_n is the *mod* p^n mapping. See [5] for some related results.

A natural approach to the Hilbert-Smith conjecture is to work on classifying spaces. Let μ^n denote the n-dimensional Menger compacta. In [1], S. M. Ageev proved that μ^n is an n-classifying space for free actions of $\widehat{\mathbb{Z}}_p$. He conjectured: for positive integers m, n and a 0-dimensional compact metric group G, if μ^{m+n} and μ^n are free G-spaces then there is no equivariant map $\mu^{m+n} \to \mu^n$. A positive way of responding to this conjecture would be to prove that there is no free p-adic group action on connected manifold X with $dim\ X/\widehat{\mathbb{Z}}_p < \infty$. Also, Edwards conjectured: the image of any $\widehat{\mathbb{Z}}_p$-equivariant map $\widehat{T}^n \to S^{\infty^\infty}(\widehat{\mathbb{Z}}_p)$ must have dimension greater

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than n. Here $S^{\infty^\infty}(\widehat{\mathbb{Z}}_p)$ is a classifying space, and a choice for $\widehat{T}^n$ is a Menger compactum. The proof for this conjecture will yield the proof for another so called *FS-ZS* conjecture. This is why mathematicians are constructing different p-adic group actions on Menger compacta.

The n-dimensional Menger compactum is universal in the sense that any n-dimensional compact metric space can be embedded in it. Thus it is a universal model for n-dimensional compacta. For instance, the zero dimensional Menger compactum is the Cantor set. Bestvina([2]) has characterized the n-dimensional Menger compactum as the unique $(m-1)$-connected and locally $(m-1)$-connected compactum having the disjoint m-cells property. There are many different constructions for Menger compacta, but only after Bestvina's 1988 paper were mathematicians able to conclude that they are all homeomorphic to each other.

Although no effective action of p-adic group on manifolds has yet been constructed, there do exist p-adic group actions on Menger manifolds. Let X be a μ^n-manifold. Then there are three constructions of $\widehat{\mathbb{Z}}_p$ actions on X.
(1) Every compact 0-dimensional metrizable group G acts effectively on X so that $dim\ X/G = dim\ X$. Hence p-adic group acts on Menger compacta. Dranishnikov [4] and Mayer and Stark [8] constructed this using Pasynkov's partial product description of μ^n. K. Sakai [10] has another construction.
(2) $\widehat{\mathbb{Z}}_p$ acts freely on X so that $dim X/\widehat{\mathbb{Z}}_p = n+1$. This example depends on the work of Bestvina, Edwards, Mayer and Stark [8].
(3) $\widehat{\mathbb{Z}}_p$ acts on X so that $dim X/\widehat{\mathbb{Z}}_p = n+2$. This was done by Mayer and Stark [8], based on a construction by Raymond and Williams [9] .

Although there are many constructions, no compact universal classifying space has yet been constructed. The goal of my thesis is to provide a more intuitive construction of p-adic group action on Menger compacta, as well as a generalization to the infinite dimensional case. Ultimately, I hope to construct a compact contractible classifying space.

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2. SOME TECHNICAL LEMMAS

Definition: Let X be a topological space, $A \subset X$ be a closed subset. Given an $\epsilon > 0$, A is called an m-ϵ absorber of X provided that for any map $f : S^k \to X$, $k \leq m$, there exits a homotopy $H : S^k \times I \to X$ such that :
(a) H is an ϵ-homotopy, i.e. for any $x_0 \in S^k$, $diam\,(Im(H(x_0, \cdot))) \leq \epsilon$,
(b) H is fixed over A, i.e. $H(x,t) = f(x)$, for any $x \in f^{-1}(A)$,
(c) H carries f into A, i.e. $H(x,0) = f(x)$, and $Im\,(H(\cdot, 1)) \subset A$.

Roughly speaking, A is an m-ϵ absorber if, for any f, there exists a homotopy pushing f into A by a small motion keeping images in A fixed.

Theorem 1. *Let R^{m+1} be partitioned into identical cubes of side 1 in the standard way. Then the m-skeleton, denoted as A, is a codimension 1, m-connected, locally m-connected subset of R^m, and it is m-$\sqrt{m+1}$ absorber.*

Note: 1. Here locally m-connectivity is in the strongest sense: every point has arbitrary small m-connected neighborhood.
2. Given any $\epsilon > 0$, for the $(m+1)$-dimensional Menger compacta, or any locally Euclidean spaces, we can construct a codimension-1 m-ϵ absorber.

If we have n copies of a compactum X, say $X_1, X_2, \cdots, X_n$, and let $A_1, A_2, \cdots, A_{m-1}$ be m-$\frac{\delta}{4n}$ absorbers of X. Let $h_i : A_i \to A_i$ be the identity map. Let $W = X_1 \cup_{h_1} X_2 \cup_{h_2} \cdots \cup_{h_{m-1}} X_m$. There is a canonical projection $\pi : W \to X$, and a canonical metric on W. For any $f : S^k \to W$, we have an $f' = \pi \circ f : S^k \to X \subset W$, such that $d(f, f') < \frac{2n\delta}{4n} = \frac{\delta}{2}$. On the other hand, by the argument in 2, f can be $\frac{n\delta}{4n}$-homotopied to some $f'' : S^k \to X$ in W. So $d(f', f'') < \frac{3n\delta}{4n} < \delta$, hence there is an ϵ-homotopy connecting f'', f' in X. Hence $f : S^k \to W$ is $(\delta + \epsilon)$-homotopic to its "projection" $f' : S^k \to X$ in W. Hence we have the following theorem which is the main tool to prove the (locally) n-connectivity of our space W.

Theorem 2. *Let W be the space constructed above. For any map $f : S^k \to W$, $k \leq n$, there exists an ϵ-homotopy H which carries f into X_1 by an ϵ-motion keeping images in X_1 fixed. Hence if X_1 is (locally) n-connected, so is X.*

3. The finite dimensional construction

Definition: Let X be a topological space, $A \subset X$ be a closed subset. Given an $\epsilon > 0$, A is called an m-ϵ absorber of X provided that for any map $f\colon S^k \to X$, $k \leq m$, there exits a homotopy $H\colon S^k \times I \to X$ such that :
(a) H is an ϵ-homotopy, i.e. for any $x_0 \in S^k$, $diam(Im(H(x_0, \cdot))) \leq \epsilon$,
(b) H is fixed over A, i.e. $H(x,t) = f(x)$, for any $x \in f^{-1}(A)$,
(c) H carries f into A, i.e. $H(x,0) = f(x)$, and $Im(H(\cdot, 1)) \subset A$.
Roughly speaking, A is an m-ϵ absorber if, for any f, there exists a homotopy pushing f into A by a small motion keeping images in A fixed.

Fact: Given an $\epsilon > 0$, for the $(m+1)$-dimensional Menger compacta, or any locally Euclidean spaces, we can construct a codimension-1 m-ϵ absorber.

For each $i = 1, 2, 3, \cdots, n+1$, let C_i be a countable dense subset of the unit interval $I = [0,1]$, such that if $i \neq j$, then $C_i \cap C_j = \emptyset$.
For any finite set $F \subset I$, we define $L(F) = \cup_{k=1}^{n} \{(x_1, x_2, \cdots, x_n) \mid x_k \in F\} \subset I^n$. If $F \subset C_i$, we define the type of F to be i.

Lemma 1. *If F is an ϵ-net of I, then $L(F)$ is an $(n-1)$-$\sqrt{n}\epsilon$ absorber of D.*

Proof. The set $L(F)$ cuts D into small cubes with diameter $< \sqrt{n}\epsilon$, hence it is an $(n-1)$-$\sqrt{n}\epsilon$ absorber of D. ☐

Lemma 2. *Given $n+1$ mutual disjoint F_i's, the intersection $\cap L(F_i) = \emptyset$.*

Proof. If $(x_1, x_2, \cdots, x_n) \in \cap L(F_i)$, then for each i, there is an $x_j \in C_i$, and $1 \leq j \leq n$. However, there are $n+1$ C_i's, $C_i \cap C_k = \emptyset$, only n x_j's. Contradiction! $\square$

For any positive geometrically decreasing sequence $\{\epsilon_k\}_{k=1}^{\infty}$, we will construct the set ${}_kL_i$ inductively.

Step 1.

For each $i \in \{1, 2, \cdots, n+1\}$, take a finite $\frac{\epsilon_1}{\sqrt{n}}$-net of C_i, denote it as ${}_1F_i$. Let ${}_1L_i = L({}_1F_i)$.

Suppose step m is constructed.

Step $m+1$.

For each $i \in \{1, 2, \cdots, n+1\}$, take a finite $\frac{\epsilon_{m+1}}{\sqrt{n}}$-net of C_i, denote it as ${}_{m+1}F_i$, such that ${}_mF_i \subset {}_{m+1}F_i$. Let ${}_{m+1}L_i = L({}_{m+1}F_i)$.

Note: Fix i, the sequence $\{{}_mL_i\}$ is an increasing sequence of closed subsets of D, each ${}_mL_i$ is a $(n-1)$-ϵ_m absorber of D.

Let $\widehat{\mathbb{Z}}_p$ be the p-adic integer group, whose underlying space is a Cantor set. Let $X = I^n \times (\widehat{\mathbb{Z}}_p)^{n+1}$. Now the group $G = (\widehat{\mathbb{Z}}_p)^{n+1}$ acts on X freely. For the p-adic integer group $\widehat{\mathbb{Z}}_p$, we use ${}_k\widehat{\mathbb{Z}}_p$ to denote the kernel of the map $\widehat{\mathbb{Z}}_p \to \mathbb{Z}/p^k\mathbb{Z}$, which is an index p^k subgroup of $\widehat{\mathbb{Z}}_p$. Let π denote the projection $X \to I^n$.

In the space X, let $(x, r_1, r_2, \cdots, r_{n+1}) \sim (x', r'_1, r'_2, \cdots, r'_{n+1})$ iff $x = x'$ and for some $k \geq 1$, one of the followings happens:

(1) $x \in {}_kL_1$, $r'_1 \in r_1 \cdot {}_k\widehat{\mathbb{Z}}_p$, and $r_i = r'_i$ for $i \neq 1$;
(2) $x \in {}_kL_2$, $r'_2 \in r_2 \cdot {}_k\widehat{\mathbb{Z}}_p$, and $r_i = r'_i$ for $i \neq 2$;
$\cdots \cdots$
(n+1) $x \in {}_kL_{n+1}$, $r'_{n+1} \in r_{n+1} \cdot {}_k\widehat{\mathbb{Z}}_p$, and $r_i = r'_i$ for $i \neq n+1$;

Denote the quotient space as W. It naturally projects to I^n.
$\widehat{\mathbb{Z}}_p$ acts on W as the diagonal subgroup of $(\widehat{\mathbb{Z}}_p)^{n+1}$.

Remark:

1. We use a positive geometrically decreasing sequence $\{\epsilon_k\}_{k=1}^{\infty}$ in the above construction. Actually, after reading the proof for the finite dimensional case, one will see that any positive sequence with a convergent sum will work for the construction.
2. On the other hand, if we choose a faster decreasing sequence, like

$$\epsilon_{k+1} = \epsilon_k / (2 \times p^{(n+1)\times(k+1)}),$$

we shall simplify the proof greatly without using the sheet-argument below.

Before we prove anything, I want to construct some subspaces W_m's and a natural metric on the space W.

The subspace W_m:

We have the following maps:

(1) If we forget the group structure, $\mathbb{Z}_{p^m} \cong \{0, 1, \cdots, p^m - 1\}$. There is a natural embedding $\mathbb{Z}_{p^m} \subset \mathbb{Z}_{p^{m+1}}$ defined by $a \mapsto a$.
(2) The projection $\phi_m \colon \mathbb{Z}_{p^{m+1}} \to \mathbb{Z}_{p^m}$ is defined by $a \mapsto a(mod\, p^m)$.
(3) The space $\widehat{\mathbb{Z}}_p$ is the inverse limit of $\{\phi_m \colon \mathbb{Z}_{p^{m+1}} \to \mathbb{Z}_{p^m}\}$. For any $m > 0$, there is a canonical map $p_m \colon \widehat{\mathbb{Z}}_p \to \mathbb{Z}_{p^m}$.
(4) On the other hand, $\widehat{\mathbb{Z}}_p$ is homeomorphic to a subgroup G of $\prod_{k=1}^{\infty} \mathbb{Z}_{p^k}$. G consists of all the elements of the form $(a_1, a_2, \cdots)$, where $a_m = \phi_m(a_{m+1})$. For any $a \in \mathbb{Z}_{p^m}$, we define $r(a) \in \widehat{\mathbb{Z}}_p$ as the unique element such that its image in G is $\widetilde{r}(a) = (a_1, a_2, \cdots)$ and $a_k = a$ for all $k \geq m$. Denote the set $\{r(a) \mid a \in \mathbb{Z}_{p^m}\}$ as A_m, hence we have an embedding $r_m \colon \mathbb{Z}_{p^m} \simeq A_m \subset \widehat{\mathbb{Z}}_p$. It induces maps $A_m \subset \widehat{\mathbb{Z}}_p \to \mathbb{Z}_{p^m} \simeq A_m$. The $(m+1)$-products of those maps naturally define:

$$D \times (A_m)^{n+1} \subset D \times (\widehat{\mathbb{Z}}_p)^{n+1} \to D \times (\mathbb{Z}_{p^m})^{n+1} \simeq D \times (A_m)^{n+1}$$

We have the following diagram:

$$\begin{tikzcd}
D \times (A_m)^{n+1} \arrow[r, hook] \arrow[d] & D \times (\widehat{\mathbb{Z}}_p)^{n+1} \arrow[r] \arrow[d] & D \times (A_m)^{n+1} \arrow[d] \\
W_m \arrow[r, hook, "i_m"] & W \arrow[r, "p_m"] & W_m
\end{tikzcd}$$

Denote projection $W \to D$ by π. For the space W_m, denote the projection from W_m to D by $\pi_m \colon W_m \to D$. W_m naturally embeds in W_{m+1}.
Take any $w \in W$, $w = \pi(x, g)$, where $x \in D$ and $g \in (\widehat{\mathbb{Z}}_p)^{n+1}$. We shall call D the horizontal direction, and $(\widehat{\mathbb{Z}}_p)^{n+1}$ the vertical direction.

There is a natural metric on W, which will simplify our proof.
(1) The space $\widehat{\mathbb{Z}}_p$ is homeomorphic to a Cantor set. It has a metric d', such that diameter of ${}_k\widehat{\mathbb{Z}}_p$ is $\frac{1}{p^k}$. Let d denote the Euclidean metric on I^n.
(2) Take any $g, g' \in G = (\widehat{\mathbb{Z}}_p)^{n+1}$, we define $d'(g, g') = \sum d'(g_i, g'_i)$, where $g = (g_1, \cdots, g_{n+1})$, $g' = (g'_1, \cdots, g'_{n+1})$.
(3) In the space $X_m = D \times (A_m)^{n+1}$, for any $x = (x_1, x_2)$, $x' = (x'_1, x'_2) \in X_m$, where $x_1, x'_1 \in D$ and $x_2, x'_2 \in (A_m)^{n+1}$, we define

$$\widetilde{d}_m(x, x') = d(x_1, x'_1) + d'(x_2, x'_2)$$

(4) The space W_m is path connected. By construction, it is a union of finite many copies of D's. For any $x, x' \in X$, there exists a piecewise linear path c connecting $p_m(x)$ and $p_m(x')$ in W_m. The path c is a union of finite many pieces of straight line segments, each lies in a copy of D. Let $l(c)$ denote the sum of lengthes of all those line segments. Let $d^*(p_m(x), p_m(x')) = \inf\{l(c)\}$ where the inf is over all possible c's connecting $p_m(x)$ and $p_m(x')$ in W_m. Define

$$d_m(p_m(x), p_m(x')) = min\{d^*(p_m(x), p_m(x')), \widetilde{d}_m(p_m(x), p_m(x'))\}$$

(5) For any $x, x' \in W$, let $d(x, x') = \underset{m}{sup}\{d_m(p_m(x), p_m(x'))\}$.
Roughly speaking, we take the product metric on $D \times (\widehat{\mathbb{Z}}_p)^{n+1}$, then make it compactible with the quotient structure on W using step (4).

4. THE (LOCALLY) $(n-1)$-CONNECTIVITY OF W_m

For any $f\colon\ S^k \to W,\ m > 0$, let $f_m = p_m \circ f\colon\ S^m \to W_m$. Let $\widetilde{f} = \pi \circ f\colon\ S^k \to D$.

Lemma 3. $d(f_m, f_{m+1}) < \frac{n+1}{p^{m+1}}$, $d(f_m, f) < \frac{n+1}{p^{m+1}}$.

Proof. For any $x \in W$, $\pi(x) = \pi(p_m(x)) = \pi(p_{m+1}(x))$ in D.

$$\begin{aligned}&d(p_m(x), p_{m+1}(x))\\ \le& sup\{\widetilde{d}_k(p_m(x), p_{m+1}(x))\}\\ =&d'(p_m(x), p_{m+1}(x))\\ <&\frac{1}{p^{m+1}} \times (n+1)\\ <&\frac{n+1}{p^{m+1}}.\end{aligned}$$

Likewise, $d(f_m, f) < \frac{2}{p^{m+1}}$. □

Definition:
W_{m+1} is a union of copies of D.
(1) When two copies D_1, D_2 of D in W_{m+1} are mapped into W_m by the map $W_{m+1} \to W_m$, if the images coincide, we say those D_1, D_2 are of the same sheet in W_m.
(2) Likewise, we also say the projections of D_1, D_2 in W_{m+1} are of the same sheet in W_m.
(3) For any two points $x, x' \in W_{m+1}$, if their projections in W_m lie in a same sheet, we also say they are of the same sheet in W_m. This is equivalent to say $p_m(x) = p_m(x')$.
(4) Moreover, take a $x \in W_{m+1}$, $x' \in W_m$, if the projection of x in W_m lies in the same sheet as x', we say x, x' are of the same sheet in W_m. □

W_m is a union of copies of D, call them $\widetilde{D}_1, \widetilde{D}_2, \cdots, \widetilde{D}_L$. W_{m+1} can be constructed from W_m by gluing copies of D, call them $D_1, D_2, \cdots, D_N$. We can reorder them such that $D_j \cap (W_m \cup D_1 \cup D_2 \cup \cdots \cup D_{j-1})$ is an $(n-1)$-ϵ_{m+1} absorber of D_j, for $j = 1, 2, \cdots, N$. The $(n-1)$-ϵ_{m+1} absorber cuts D_j into small cubes of diameter $< \epsilon_{m+1}$, and $D_1, \cdots D_{N_1}$ are of the same sheet with $\widetilde{D}_1$ in W_m; $D_{N_1+1}, \cdots, D_{N_2}$ are of the same sheet with $\widetilde{D}_2$ in W_m; $\cdots$.

Lemma 4. *Let $A_j = W_m \cup D_1 \cup D_2 \cup \cdots \cup D_{j-1}$. For any $f\colon\ S^k \to W_{m+1}$, $k < n$, there is a homotopy H carries f to $f_j\colon\ S^k \to A_j$, and H is fixed on A_j. Moreover, $d(\pi \circ H(\cdot, t), \pi \circ f) < \epsilon_{m+1}$ for any $t \in I$, and for any $x \in W_{m+1}$, $f(x), f_j(x)$ are of the same sheet in W_m.*

Proof. Use reverse induction.
For $j = N$, this is clear from the ordering of D_i.
If this is true for $j = l+1$, then there is a homotopy H_1, which carries $f\colon\ S^k \to W_{m+1}$ to $f_{n+1}\colon\ S^k \to A_{l+1}$, and is fixed on A_{l+1}, with $d(\pi \circ H(\cdot, t), \pi \circ f) < \epsilon_{m+1}$ for any $t \in I$, and for any $x \in W_{m+1}$, $f(x), f_{l+1}(x)$ are of the same sheet in W_m.

Now for the case $j = l$, $A_{l+1} = A_l \cup D_l$, where $A_l \cap D_l$ is an $(n-1)$-ϵ_{m+1} absorber of D_l. Denote $B_l = S^k - f^{-1}(A_l)$. Since $f(B_l) \subset D_l$, and $A_l \cap D_l$ is an $(n-1)$-ϵ_{m+1} absorber of D_l, and cuts D_l into small cubes of diameter $< \epsilon_{m+1}$, $f_{l+1}|_{A_l}$ can be extended to an $f_l \colon S^k \to A_l$, such that $d(\pi \circ f_l, \pi \circ f) < \epsilon_{m+1}$, and for any $x \in W_{m+1}$, $f(x), f_l(x)$ are of the same sheet in W_m. D is a convex set, for any $x \in B_l$, there is a strait line segment in D from $f_{l+1}(x)$ to $f'(x)$. This defines a homotopy H_2 from f_{l+1} to f'. It is clear that $d(\pi \circ H_2(\cdot, t), \pi \circ f) < \epsilon_{m+1}$ for any $t \in I$. Then the composition of H_2 and H_1 is the desired homotopy. Call it H. It has the same property: $d(\pi \circ H(\cdot, t), \pi \circ f) < \epsilon_{m+1}$ for any $t \in I$, hence it is a $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopy. □

Corollary 4.1. *A_j is an $(n-1)$-$(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$ absorber of W_{m+1}. In particular, W_m is an $(n-1)$-$(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$ absorber of W_{m+1}.*

Repeat applying the above argument, f_{m+1} can be homotopied to an $f'_m \colon S^k \to W_m$ by a $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopy, and π

Lemma 5. *The distance $d(f_m, f'_m) < \epsilon_{m+1}$, and f_m, f'_m can be ϵ_{m+1}-homotopied to each other. Hence f_m, f_{m+1} are $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopic to each other.*

Proof. For any $x \in W_{m+1}$, $f_{m+1}(x), f'_m(x)$ are of the same sheet in W_m, on the other hand $f_m(x), f_{m+1}(x)$ are of the same sheet in W_m, then $f_m(x), f'_m(x)$ are of the same sheet in W_m. On the other hand, $d(\pi \circ f_m = \pi \circ f_{m+1}, \pi \circ f) < \epsilon_{m+1}$, hence $d(f_m, f'_m) < \epsilon_{m+1}$.
Since $f_m(x), f'_m(x)$ are of the same sheet in W_m, there is a unique line segment from $f_m(x)$ to $f'_m(x)$ in W_m. This defines an ϵ_{m+1}-homotopy between them. □

Lemma 6. *The space W is $(n-1)$-connected and locally $(n-1)$-connected.*

Proof. f_m, f_{m+1} are $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopic to each other. Likewise, we can prove that for any $k > 0$, f_m, f_{m+k} are $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopic to each other. On the other hand, $\lim f_m = f$. Hence f, f_m are $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1})$-homotopic to each other. By the lemmas from the section of technical lemmas, we know W_m is $(n-1)$-connected. Hence W is $(n-1)$-connected.
For the locally $(n-1)$-connectivity, notice W_m is locally $(n-1)$-connected and compact, so for any $\epsilon > 0$, there is a $\delta_m > 0$, such that any map $f \colon S^k \to W_m$ with $diam(Im(f)) < \delta_m$ is null homotopic by an ϵ-homotopy. Then for any $\epsilon > 0$, there is an $m > 0$, such that $(\frac{n+1}{p^{m+1}} + \epsilon_{m+1}) < \epsilon$. Then for any $f \colon S^k \to W$ with $diam(Im(f)) < \delta_m$, we have $diam(Im(f_m)) \leq diam(Im(f)) < \delta_m$. f can be ϵ-homotopied to f_m, and f_m is null homotopic by an ϵ-homotopy, hence f is null homotopic by a 2ϵ-homotopy. □

5. DIMENSION

Definition The *Urysohn n-diameter* of a space X is the infimum $d_n(X)$ of all positive numbers ϵ such that there is a an ϵ-map $f \colon X \to Y$ for some n-dimensional space Y.

Theorem 3. *([3]) A compact metric space X has Urysohn n-diameter 0 if and only if $dimX \leq n$.*

We have the ϵ-map $p_m \colon W \to W_m$, where W_m is n-dimensional, $\epsilon = \epsilon_m \to 0$. Hence the quotient space W we construct here has *Urysohn n-diameter* 0, so $dimW \leq n$. On the other hand, $W_m \subset W$, so $dimW \geq n$. Hence $dimW = n$.

6. The disjoint m-cells property

There are two ways to prove the disjoint m-cells property, one is to use a Z-set argument, another is a short cut:

$$\begin{tikzcd}
D \times (A_m)^{n+1} \arrow[r, hook] \arrow[d] & D \times (A_{m+1})^{n+1} \arrow[r, hook] \arrow[d] & D \times (\widehat{\mathbb{Z}}_p)^{n+1} \arrow[d] \\
W_m \arrow[r, hook] & W_{m+1} \arrow[r, hook] & W
\end{tikzcd}$$

$W_m \subset W_{m+1} \subset W$, and there are $p^{(n+1)}$ copies of W_m in W_{m+1}. The group $(\mathbb{Z}_{p^{m+1}})^{n+1}$ acts on the space $D \times (A_{m+1})^{n+1}$. This induces an effective action of $(\mathbb{Z}_{p^{m+1}})^{n+1}$ on W_{m+1}. Notice W_m and $(1, 1, \cdots, 1) * W_m$ are disjoint from each other in W_{m+1}, so are they in W. On the other hand, there is a natural projection $p_m \colon W \to W_m$. Let $p'_m = (1, 1, \cdots, 1) * p_m \colon W \to (1, 1, \cdots, 1) * W_m$. For any maps $f_1, f_2 \colon S^n \to W$, let $f'_1 = p_m \circ f_1$, $f'_2 = p'_m \circ f_2$. Then f_1 and f_2 have disjoint images, and $d(f_1, f'_1)$ and $d(f_2, f'_2)$ can be arbitrarily small if m is large enough.

Corollary: The space W is homeomorphic to the n-dimensional Menger compactum.

Remark: This construction can generalized to infinite dimension. One version gives a space W_1 which is compact, n-connected and locally n-connected for any $n > 0$. Another version gives a space W_2 which is compact, contractible, locally contractible. However, for the space W_2, it only supports an effective p-adic group action, not a free action.

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