

## ON TRIPLE POINT NUMBERS OF SURFACE-KNOTS

TSUKASA YASHIRO

ABSTRACT. A surface-knot is an embedded oriented closed connected surface in 4-space. The triple point number of a surface-knot is the minimal number of triple points for all generic projections of the surface-knot. The present paper discusses about sub-diagrams of a surface-knot, in which some triple points can be eliminated by finite sequences of local moves. This gives an estimate of the triple point number of the surface-knot.

### 1. INTRODUCTION

A *surface-knot* is an embedded closed connected oriented surface in 4-space. If the surface is the sphere, then it is called a *2-knot*. Let  $\rho : \mathbf{R}^4 \rightarrow \mathbf{R}^3$  be the projection defined by  $\rho(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$ . The singularity of a surface-knot  $F$  is the set  $\tilde{S}(F) = \{x \in F : |F \cap \rho^{-1}(\rho(x))| \geq 2\}$ . We denote the image  $\rho(\tilde{S}(F))$  by  $S(\rho)$ . For  $x \in \tilde{S}(F)$ ,  $\rho(x)$  is called a *double point* if  $|F \cap \rho^{-1}(\rho(x))| = 2$  and is called a *triple point* if  $|F \cap \rho^{-1}(\rho(x))| = 3$ . The set of double points is denoted by  $S_2(\rho)$  and is an open 1-dimensional manifold. The set of triple points is denoted by  $S_3(\rho)$ . A point  $y$  of  $\rho(F)$  is called a *branch point* if  $y$  has an open 3-ball neighbourhood  $B(y)$  of  $y$  in  $\mathbf{R}^3$ , where  $B(y) \cap \rho(F)$  is homeomorphic to an open cone over the figure eight. The closure of each component of  $S_2(\rho)$  is called a *double segment*, which is bounded by triple points or branch points. The orientation of  $S(\rho)$  is determined by the orientation of  $D_F$  [Sat1].

Assume that there exist discs  $E$ ,  $E_0$  and  $E_1$  on  $F$  such that  $\rho(E) \cap \rho(E_0)$  and  $\rho(E) \cap \rho(E_1)$  form a pair of double arcs  $\gamma_0$  and  $\gamma_1$  on  $\rho(E)$  respectively. If there exist a simple arc  $\gamma$  on  $\rho(E)$  from a point of  $\gamma_0$  to a point of  $\gamma_1$  such that  $\text{Int}(\gamma) \cap S(\rho) = \emptyset$  and orientations of  $\gamma_0$  and  $\gamma_1$  at ends of  $\gamma$  are opposite with respect to the orientation of  $\rho(E)$ , then we call the pair of arcs  $\{\gamma_0, \gamma_1\}$  a *cancelling pair* of double arcs.

The projection  $\rho(F)$  is called a *generic projection* if for every point  $x \in \mathcal{S}(F)$ ,  $\rho(x)$  is either a double point or a triple point or a branch point (see [Yaj][CS1][Kaw]). A *surface diagram* of a surface-knot is a generic projection of the surface-knot with height information. A triple point is formed by three sheets; called the *top sheet*, the *middle sheet* and the *bottom sheet* with respect to the last coordinate of  $\mathbf{R}^4$  (see [CS1]). Let  $F$  be a surface-knot and let  $D_F$  be a surface diagram of  $F$ . Then we will denote the number of triple points of  $D_F$  by  $t(D_F)$ . The *triple point number* of a surface-knot is the minimal number of triple points for all surface diagrams of the surface-knot. We denote this by  $t(F)$ . A 2-knot  $F$  with  $t(F) = 0$  is a ribbon 2-knot [Yaj]. A surface-knot  $F$  with  $t(F) = 0$  is called a *pseudo ribbon 2-knot* (see [Kaw]). There is no 2-knot  $F$  with  $t(F) = 1$  due to S. Satoh. A 2-knot with  $t(F) = 2$  is a ribbon 2-knot [Sh]. The triple point number of  $q$ -twist spun trefoil is 4 if  $q = 2$  and is 6 if  $q = 3$  [SS] and the triple point number of  $q$ -twist-spin

of a knot  $K$  is bounded above by  $2q(c(K) - 1)$  where  $c(K)$  is the crossing number of  $K$  (Theorem 3.1).

We obtain an estimate of the triple point number of a surface-knot  $F$  by deforming its surface diagram  $D_F$  into a simpler one;  $D'_F$  with less triple points. Let  $p$  be a point of  $D_F$  and let  $B(p)$  be a closed 3-ball centred at  $p$  in  $\mathbf{R}^3$ . A *sub-diagram* of  $D_F$  is a pair  $(B(p) \cap D_F, \partial(B(p) \cap D_F))$ . We will denote this by  $(T, \partial T)$ . Since  $(T, \partial T)$  has height information,  $\partial T$  is a link diagram in  $\partial B(p)$ . Note that  $\partial T$  is not necessary to be trivial. In the following, we will focus on sub-diagrams  $(T, \partial T)$  consisting of proper discs in  $B(p)$  and trivial  $\partial T$  as a link diagram in  $\partial B(p)$ . In Section 3, we will construct some sub-diagrams, in which we can eliminate some triple points.

2. ROSEMAN MOVES

Reidemeister moves (see [Rol]) consist of three types of local deformations for a knot diagram. We have a generalization of Reidemeister moves for surface-knots

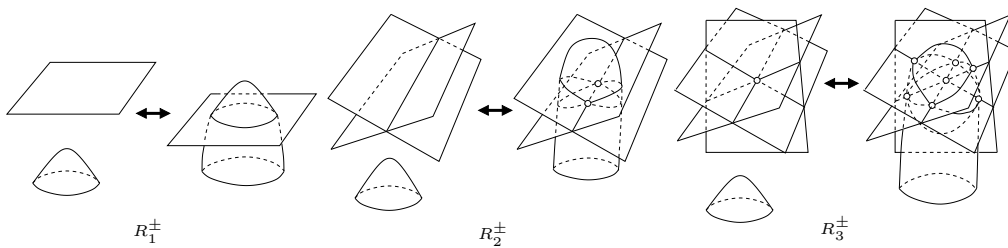


FIGURE 1

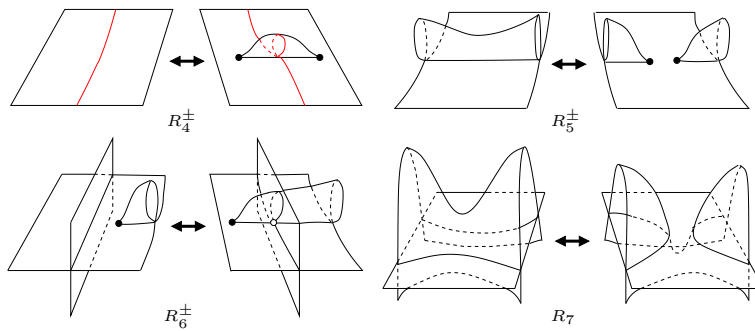


FIGURE 2

as the following.

**Theorem 2.1** (Roseman[Ros]). *If equivalent surface-knots  $F_1$  and  $F_2$  are given, then a diagram of  $F_1$  is deformed into a diagram of  $F_2$  by a finite sequence of seven types of local deformations.*

Those deformations in the above theorem are called *Roseman moves* depicted in Figure 1 and Figure 2. Note that  $R_2^-$ ,  $R_3^-$  and  $R_5^-$  moves eliminate some triple points in a surface diagram.

## 3. SUB-DIAGRAMS

In this section, we will construct some sub-diagrams of a surface diagram. The sub-diagram will have certain property; that is, some triple points in those sub-diagrams can be eliminated by a finite sequence of local moves.

In the following, we will denote the  $xy$ -plane, the  $yz$ -plane and the  $xz$ -plane by  $P_{xy}$ ,  $P_{yz}$  and  $P_{xz}$  respectively. Let  $t_0$  denote the origin of  $\mathbf{R}^3$  and let  $B(t_0)$  denote the closed unit 3-ball in  $\mathbf{R}^3$  centred at  $t_0$ .

First, we will see an elimination of a triple point by  $R_6^-$  move. Obviously, this is a deformation of a sub-diagram. We denote the sub-diagram by  $(T(2), \partial T(2))$ , where  $T(2)$  consists of a disc and a disc with a branch point (see Figure 2). Then the following holds.

**Lemma 3.1.** *Let  $F$  be a surface-knot and let  $D_F$  be a surface diagram. If  $D_F$  has  $k$  copies of the sub-diagram  $(T(2), \partial T(2))$ , then  $t(F) \leq t(D_F) - k$ .*

*Proof.* The result is an immediate consequence of the definition of the sub-diagram  $(T(2), \partial T(2))$ .  $\square$

The following holds.

**Lemma 3.2.** *Let  $F$  be the  $q$ -twist-spin of a knot  $K$ . Then there exists a surface diagram  $D_F$  with  $q$  copies of  $(T(2), \partial T(2))$  such that  $t(D_F) = 2q(c(K))$ , where  $c(K)$  is the crossing number of  $K$ .*

Above two lemmas imply the following.

**Theorem 3.1** (Satoh [Sat2]). *Let  $F$  be the  $q$ -twist-spin of a knot  $K$ . Then  $t(F) \leq 2q(c(K) - 1)$ .*

We omit proof of Lemma 3.2 and Theorem 3.1.

Second, we consider an elimination of a pair of triple points in a surface diagram. If a pair of triple points at the ends of a double edge satisfies certain condition, then those triple points can be eliminated by a finite sequence of Roseman moves.

Let  $t_0$  denote the origin of  $\mathbf{R}^3$  and let  $B(t_0)$  be the closed unit 3-ball in  $\mathbf{R}^3$ . Let  $P_+$  and  $P_-$  be planes  $x = 1/2$  and  $x = -1/2$  respectively. Set  $E_{\pm} = P_{\pm} \cap B(t_0)$ ,  $D_1 = P_{xz} \cap B(t_0)$  and  $D_2 = P_{xy} \cap B(t_0)$ . Let  $E_0$  be a band on  $\partial B(t_0)$  homeomorphic to  $I \times I$  so that  $E_0 \cap E_+ \cong \{0\} \times I$  and  $E_0 \cap E_- \cong \{1\} \times I$ . After smoothing the corners,  $E_0 \cup E_+ \cup E_-$  is a disc and we will denote it by  $\widehat{E}$ . Push the interior of  $E_0$  into  $B(t_0)$  so that  $\widehat{E}$  becomes a proper disc in  $B(t_0)$ . Add height information to lift the diagram into  $\mathbf{R}^4$  so that the relative heights of  $E_+$  and  $E_-$  against heights of  $D_1$  and  $D_2$  are the same. We will denote the diagram by  $(T(3), \partial T(3))$ . Then the following holds.

**Theorem 3.2.** *Let  $F \subset \mathbf{R}^4$  be a surface-knot and let  $D_F$  be a surface diagram in  $\mathbf{R}^3$ . Then if  $D_F$  contains  $k$  copies of  $(T(3), \partial T(3))$ , then  $t(F) \leq t(D_F) - 2k$*

*Proof.* Without loss of generality, we can assume that  $E_0 \cap (D_1 \cup D_2) = \emptyset$  and  $E_0$  is in the region  $y > 0, z > 0$ . Assume that there are two arcs on  $E_0$ ;  $I \times \{1/4\}$  and  $I \times \{3/4\}$  denoted by  $\delta_1$  and  $\delta_2$  respectively. Then project  $\delta_1$  onto the image

$\delta'_1$  in  $D_1$  so that there is a disc  $E_1$  bounded by  $\delta_1$ ,  $\delta'_1$  and arcs between  $\partial\delta_1$  and  $\partial\delta'_1$ . It is not too difficult to see that  $E_1$  is a guide-disc for  $R_7$ -move. Apply the Roseman move to create a pair of double curves on  $D_1$ , one of which joins  $t_1^+$  and  $t_1^-$ . Similarly, we can project  $\delta_2$  onto  $\delta'_2$  in  $D_2$ . We obtain a pair of double arcs on  $D_2$  as the above. Now  $R_2^-$ -move can be applied to the diagram to eliminate those two triple points.  $\square$

**Example 3.3.** Let  $K$  be the trefoil and let  $F$  be the  $q$ -twist-spin of  $K$ , ( $q \geq 2$ ). Then there exists a surface diagram  $D_F$  of  $F$  such that  $t(D_F) = 6q$  and  $D_F$  contains  $2q$  copies of  $(T(2), \partial T(2))$  and  $q$  copies of  $(T(3), \partial T(3))$  [Sat2][Yas2]. Therefore,  $t(F) \leq t(D_F) - 4q = 2q$ .

Next, we will construct a model containing seven triple points, in which six of them can be eliminated by a finite sequence of Roseman moves. Then we will construct another model containing some triple points, in which all triple points can be eliminated by a finite sequence of Roseman moves.

Set  $D_1 = B(t_0) \cap P_{xz}$ ,  $D_2 = B(t_0) \cap P_{xy}$  and  $D_3 = B(t_0) \cap P_{yz}$ . Let  $P_+$  be the plane  $x + y + z = 1/2$  and let  $P_-$  be the plane  $x + y + z = -1/2$ . Set  $E_\pm = P_\pm \cap B(t_0)$ ,

- (1)  $t_1^+ = (1/2, 0, 0), \quad t_1^- = (-1/2, 0, 0),$
- (2)  $t_2^+ = (0, 1/2, 0), \quad t_2^- = (0, -1/2, 0),$
- (3)  $t_3^+ = (0, 0, 1/2), \quad t_3^- = (0, 0, -1/2).$

Take a disc  $E_0$  on the boundary  $\partial B(t_0)$  so that  $E_0$  is homeomorphic to  $I \times I$ ,  $E_0 \cap \{0\} \times I$  and  $E_0 \cap E_- = \{1\} \times I$ ,  $E_0 \cap (D_1 \cup D_2 \cup D_3) = \emptyset$ . We denote the union  $E_+ \cup E_0 \cup E_-$  by  $\widehat{E}$  and denote the set  $\widehat{E} \cup (D_1 \cup D_2 \cup D_3)$  by  $Q$ .

Now we assume that  $Q$  has height information on  $Q$  so that three discs  $D_1, D_2$  and  $D_3$  are lifted in  $\mathbf{R}^4$  as disjoint discs. We will denote the diagram by  $(T(4), \partial T(4))$ . Then  $E_+$  and  $E_-$  contain triangles formed by double segments and triple points,  $\{t_1^+, t_2^+, t_3^+\}$  and  $\{t_1^-, t_2^-, t_3^-\}$ .

We have the following.

**Theorem 3.4.** *Let  $F$  be a surface-knot and let  $D_F$  be a surface diagram of  $F$ . Assume that there are  $k$  copies of  $(T(4), \partial T(4))$  in  $D_F$ . Then  $t(F) \leq t(D_F) - 6k$ .*

*Proof.* We will deform  $T(4)$  instead of deforming the corresponding sub-diagram of  $D_F$ . We will show that the deformation eliminates triple points  $\{t_1^+, t_2^+, t_3^+\}$  and  $\{t_1^-, t_2^-, t_3^-\}$ . This deformation shows that six triple points in  $T(4)$  can be eliminated by a finite sequence of Roseman moves.

Let  $\mathbf{n}_0$  be the vector  $(1, 1, 1)$ . Divide  $\mathbf{R}^3$  into two pieces with respect to  $\mathbf{n}_0$ ;  $\mathbf{R}_+^3 = \{x \in \mathbf{R}^3 \mid x \cdot \mathbf{n}_0 \geq 0\}$ ,  $\mathbf{R}_-^3 = \{x \in \mathbf{R}^3 \mid x \cdot \mathbf{n}_0 < 0\}$ , where  $x \cdot \mathbf{n}_0$  means the inner product.

Push the plane  $P_+$  along the vector  $\mathbf{n}_0$  through  $t_0$  into  $\mathbf{R}_-^3$  so that the moved plane  $P'_+$  is between  $P_-$  and the triple point  $t_0$ .  $B(t_0) \cap P'_+$  will be denoted by  $E'_+$ . There is an isotopy deformation moving  $E_+$  onto  $E'_+$  in  $\mathbf{R}^4$ . The band  $E_0$  is also deformed into  $E'_0$  on  $\partial B(t_0)$  joining  $E_-$  and  $E'_+$ . Note that this deformation can be described by  $R_2^+$  moves,  $R_7$  moves and  $R_3^-$  moves. We will use the same notation  $t_1^+, t_2^+, t_3^+$  for the corresponding moved triple points.

Assume that  $E'_0 \subset \partial B(t_0)$  has a core arc  $\gamma_1 = I \times \{1/2\}$ , and assume that there is an arc  $\gamma_2$  between end points of  $\gamma_1$  on  $E'_+ \cup E_- \cup D_1 \cup D_2 \cup D_3$ . Then it is not difficult to see that  $\gamma_1 \cup \gamma_2$  bounds a disc  $E$  in  $B(t_0)$  so that  $Int(E) \cap T(4) = \emptyset$ .

Note that  $\partial E$  consists of two arcs  $\gamma'_1$  and  $\gamma'_2$  so that these connect a cancelling pair of double edges. Without loss of generality we can assume that the edges containing the cancelling pair are  $t_1^+ t_2^+$  and  $t_1^- t_2^-$ . Since the disc  $E$  is a guide-disc, we can apply  $R_7$  move so that those edges will be joined. As the result of the deformation, new arcs  $\delta_1$  and  $\delta_2$  join pairs of triple points  $\{t_1^+, t_1^-\}$  and  $\{t_2^+, t_2^-\}$  respectively. Then the arc  $\delta_i$  and edge  $t_i^+ t_i^-$  bounds a disc, which is a guide-disc for each  $i = 1, 2$ . From Theorem 3.2 those pairs of triple points  $\{t_1^+, t_1^-\}$  and  $\{t_2^+, t_2^-\}$  can be eliminated. After this deformation  $t_3^+$  and  $t_3^-$  are on the boundary of another guide-disc so that they can be eliminated. This completes the proof.  $\square$

Finally, we will construct a sub-diagram with some triple points and a pair of branch points. Let  $D^2(O; 2)$  be a closed disc in  $\mathbf{R}^2$  with radius 2 centred at the origin  $O$ . Take  $4m$  points on  $\partial D^2(O; 2)$ ;

$$(4) \quad p_k = \left( 2 \cos \frac{\pi k}{m}, 2 \sin \frac{\pi k}{m} \right),$$

$$(5) \quad q_k = \left( 2 \cos \left( \frac{\pi k}{m} + \frac{\pi}{2m} \right), 2 \sin \left( \frac{\pi k}{m} + \frac{\pi}{2m} \right) \right),$$

where  $p_0 = p_{2m}$ ,  $q_0 = q_{2m}$  and  $k = 0, \dots, 2m - 1$ . Take  $2m$  chords  $p_k q_{k+1}$ , ( $k = 0, \dots, 2m - 1$ ).

Set the following notations.

$$(6) \quad c_k = p_k q_{k+1} \cap p_{k-1} q_k,$$

$$(7) \quad e_k = c_k c_{k+1}, \quad (k = 0, \dots, 2m - 1),$$

$$(8) \quad F = \bigcup_{k=0}^{2m-1} p_k q_{k+1}.$$

Note that  $\cup_{k=0}^{2m-1} e_k$  forms a  $2m$ -gon in  $D^2(O; 2)$ . Let  $J$  be the closed interval  $[0, n+1]$ , ( $n \geq 1$ ).

$$(9) \quad E_i = D^2(O; 2) \times \{i\}, \quad (i = 1, \dots, n),$$

$$(10) \quad Q = F \times J \cup \left( \bigcup_{i=1}^n E_i \right),$$

$$(11) \quad c_k^i = c_k \times \{i\}, \quad (0 \leq k \leq 2m - 1, 1 \leq i \leq n),$$

$$(12) \quad e_k^i = e_k \times \{i\}, \quad (0 \leq k \leq 2m - 1, 1 \leq i \leq n).$$

Note that  $c_k^i$  are triple points.

We will add height information to  $Q$  so that the constructed diagram can be lifted into 4-space. By adding height information to double edges, we can orient them so that the pair  $\{e_k^i, e_{2m-k}^i\}$  is a cancelling pair.

Assume that triple points,  $c_0^n, c_m^n$  connects to a branch point by double segments  $\gamma_0$  and  $\gamma_m$  respectively. Put trivial bands  $E_{k, 2m-k}$  to join  $e_k \times (n+1)$  and  $e_{2m-k} \times (n+1)$ , ( $0 \leq k \leq 2m - 1$ ).

Now we add an extra crossing information to the diagram as the following. For each  $i = 1, \dots, n$ , fix  $k \neq 0, m$ .  $E_i$  is the top sheet at the triple point  $c_j^i$ , ( $j = k + 1, \dots, 2m - k - 1$ ),  $E_i$  is the middle sheet at  $c_k^i$  and  $E_i$  is the bottom sheet at each of other triple points. Note that we have constructed a diagram consisting of

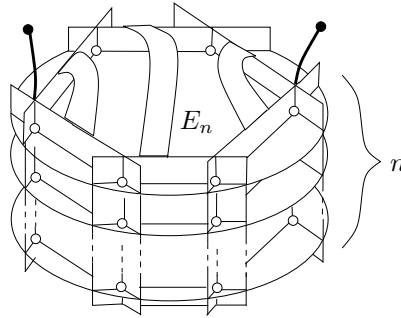


FIGURE 3.  $(T(3+n), \partial T(3+n))$

$m+n$  discs. The boundary of the diagram is an  $(m+n)$ -component link diagram, which is determined by the above height information. We denote this diagram by  $(T(m+n), \partial T(m+n))$ , (see Figure 3 for  $(T(3+n), \partial T(3+n))$ )

**Theorem 3.5.** *Let  $F$  be a surface-knot and let  $D_F$  be a surface diagram of  $F$ . If the surface diagram  $D_F$  contains  $k$  copies of the sub-diagram  $(T(m+n), \partial T(m+n))$ , then  $t(F) \leq t(D_F) - 2kmn$ .*

*Proof.* We can find guide-discs at bands  $E_{k, 2m-k}$  so that  $R_7$ -moves can be applied. Then from Theorem 3.2 and 3.4, we can eliminate triple points  $c_k^n$ , ( $0 \leq k \leq 2m-1$ ). It is not too difficult to see that the next stage has the same situation so that we can repeat the same operation to the stage and repeat this until  $2mn$  triple points are eliminated. We can apply the same operation to each of other sub-diagrams to eliminate all triple points in those sub-diagrams. This completes the proof.  $\square$

The following holds.

**Theorem 3.6.** *Let  $K$  be the  $(2, k)$ -torus knot with odd  $k \geq 3$  and let  $F$  be the  $q$ -twist-spin of  $K$  with  $q \geq 2$ . Then there exists a surface diagram  $D_F$  of  $F$  such that  $t(D_F) = 3q(k-1)$  and  $D_F$  contains  $q$  copies of  $(T(2 + \frac{k-1}{2}), \partial T(2 + \frac{k-1}{2}))$ .*

We omit the proof. This theorem implies the following.

**Corollary 3.1** ([Yas2]). *Let  $F$  be the  $q$ -twist-spin of a  $(2, k)$ -torus knot with odd  $k \geq 3$ , ( $q \geq 2$ ). Then  $t(F) \leq q(k-1)$ .*

## REFERENCES

- [CS1] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*. Mathematical Surveys and Monographs, 55. American Mathematical Society, Providence, RI, 1998.
- [CS2] J. S. Carter and M. Saito, "Surfaces in 3-space that do not lift to embeddings in 4-space" *Knot theory Banach center publications*, **42** Warszawa (1998), 29-47.
- [HK] F. Hosokawa, A. Kawachi, "Proposals for unknotted surfaces in four-spaces", *Osaka J. Math.* **16** (1979), 233-248.

- [Hom] T. Homma, "Elementary deformations on Homotopy 3-Spheres", *Topology and Computer Science*, 21-27, Kinokuniya, Tokyo, 1987.
- [Kaw] A. Kawauchi, "On pseudo-ribbon surface-links", *J. Knot Theory and its Ram.* **11**, No. 7 (2002) 1043-1062.
- [Rol] D. Rolfsen, "*Knots and links*", Mathematics Lecture Series No. 7, Publish or Perish Inc., Berkeley, Calif., 1976.
- [Ros] D. Roseman, "Reidemeister-type moves for surfaces in four dimensional space", *Banach Center Publications* **42** (1998), *Knot Theory*, 347-380.
- [Sat1] S. Satoh, "Lifting a generic surface in 3-space to an embedded surface in 4-space" *Topology Appl.* **106**, No. 1 (2000) 103-113.
- [Sat2] S. Satoh, "Surface diagrams of twist-spun 2-knots", *J. Knot Theory and its Ram.* **11**, No. 3 (2002) 413-430.
- [Sh] A. Shima, "Knotted 2-spheres whose projections into 3-space contain at most two triple points are ribbon 2-knots", preprint.
- [SS] S. Satoh and A. Shima, "Triple point numbers of surface knots and colorings by quandles", preprint.
- [Yaj] T. Yajima, "On simple knotted spheres in  $\mathbf{R}^4$ ", *Osaka J. Math.* **1** (1969), 133-152.
- [Yas1] T. Yashiro, "Immersions and their lifts", *NZ J. Math.*, **30** (2001), 197-210.
- [Yas2] T. Yashiro, "Triple point numbers of twist-spun knots", preprint.
- [Zee] E. C. Zeeman, "Twisting spun knots", *Trans. Amer. Math. Soc.*, **115** (1965), 471-495.

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 3-3-138, SUGIMOTO, SUMIYOSHI, OSAKA, JAPAN