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FIXED POINTS OF SOME HOMEOMORPHISMS ON 3-DIMENSIONAL REDUCIBLE MANIFOLDS

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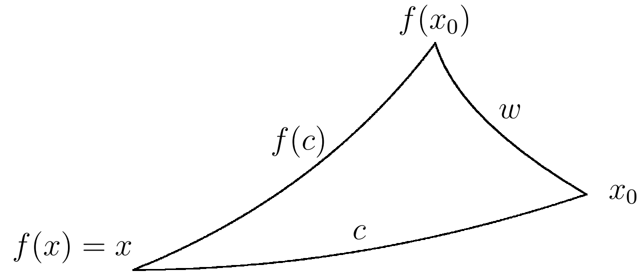
Abstract. In this talk, we consider the fixed points of some homeomorphisms of closed oriented reducible 3-manifolds. Up to an isotopy, we compute out the fixed point indices and fixed point coordinates for all isolated fixed point sets of homeomorphisms composed by two slides homeomorphism. An upper bound for the Nielsen numbers of these homeomorphims is obtained. Finally, we shall show by some examples that Nielsen numbers of such kind of homeomorphisms can be arbitrarily large, although their Lefschetz numbers are zero.

1. What is the Nielsen theory doing?

The main problem in Nielsen fixed point theory is to estimate the number of fixed point of a given map $f\colon X \to X$. In a general case, X may be a compact polyhedron. Here, we talk about the case X is a connected compact manifold. Basic materials of Nielsen fixed point theory can be found in [1], [3] or [7].

(1.1) The coordinate of a fixed point.

Pick a base point $x_0 \in X$ and a path w from x_0 to $f(x_0)$. Any fixed point x of f, together with a path c from x_0 to x, corresponds to an element $\langle c(f(c))^{-1}w^{-1}\rangle$ in $\pi_1(X, x_0)$.



In $\pi_1(X, x_0)$, two elements α and β are said to be f-conjugate if there exists γ such that

$$\alpha = \gamma^{-1}\beta w_*(f_\pi(\gamma)),$$

where $f_\pi\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is the homomorphism induced by f, and the isomorphism $w_*\colon \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ is given by $w_*(\langle a\rangle) = \langle waw^{-1}\rangle$.

Proposition 1.1. *Each fixed point of f corresponds to a unique f-conjugacy class in $\pi_1(X, x_0)$.*

(1.2) Fixed point class.

Two fixed points are said to be in the same fixed point class if and only if they have the same coordinates. So, we also call f-conjugacy classes as fixed point classes. There may be several "different" empty fixed point classes, because they are (determined by) different f-conjugacy classes.

(1.3) fixed point index.

For an isolated point x, the index of x is $sgn(det(I - df|_x))$ if it is non-zero.

Proposition 1.2. *The sum of indices all fixed points of f is the Lefschetz number $L(f)$.*

Proposition 1.3. *The fixed point index of any empty fixed point class is zero.*

Proposition 1.4. *Each fixed point class is an isolated fixed point set, and hence has an index.*

(1.4) Homotopy invariance

There is a one-to-one correspondence between sets of fixed point classes of homotopic maps.

Homotopy-related fixed point classes have the same fixed point indices.

(1.5) Nielsen number

The number of fixed point classes of f with non-zero indices (essential classes), denoted $N(f)$.

Theorem 1.5. *(Nielsen 1927; Reidemeister-Wecken 1941) Any map homotopic to f has at least $N(f)$ fixed points.*

(1.6) An example

Example 1.6. *Let $X = \{e^{\theta i}\}$ be a circle. A self map $f\colon X \to X$ is defined by $f(e^{\theta i}) = e^{-\theta i}$. It has two fixed points $\{e^0, e^{\pi i}\}$.*

Take e^0 as the base point x_0 in X and the constant path at e^0 as base path. Two elements α and β in $\pi_1(X, x_0)$ are f-conjugate if there exists an element $\gamma \in \pi_1(X, x_0)$ such that $\alpha = \gamma^{-1}\beta w_*(f_\pi(\gamma))$, it means $\alpha = \gamma^{-1}\beta f_\pi(\gamma) = \gamma^{-1}\beta\gamma^{-1} = \beta\gamma^2$.

Thus, the set of f-conjugacy classes have two elements, which are represented by 1 and the generator a of $\pi_1(X, e^0) = Z$.

The coordinate of the fixed point e^0 is 1, and the coordinate of the fixed point $e^{\pi i}$ is

$$\begin{aligned} & \langle \{e^{t\pi i}\}_{0\le t\le 1}(f(\{e^{t\pi i}\}_{0\le t\le 1}))^{-1}\rangle \\ = & \langle \{e^{t\pi i}\}_{0\le t\le 1}(\{e^{-t\pi i}\}_{0\le t\le 1})^{-1}\rangle \\ = & a \end{aligned}$$

Here a path c from base point e^0 to $e^{\pi i}$ is chosen as $\{e^{t\pi i}\}_{0\le t\le 1}$.

2. A question: Is the Nielsen number $N(f)$ optimal?

In the case that f is a self map, we may ask such a question: Is there a map g homotopic to f with $|\mathrm{Fix}(g)| = N(f)$?

When $dimX \ge 3$, the answer is "YES" (Wecken 1942 [9]).

When $dimX = 2$, the answer is "NO" ([2]) if $\chi(X) < 0$; "YES" otherwise.

In the case that f is a homeomorphisms, we may ask a similar question: Is there a homeomorphism g isotopic to f with $|\mathrm{Fix}(g)| = N(f)$?

When $dimX = 2$, the answer is "YES" (Jiang and Guo, 1993 [4]).

When $dimX \ge 5$, the answer is "YES" (Kelly, 1995 [6]).

When $dimX = 3$, the answer is "YES" (Jiang, Wang and Wu [5]) if X is closed oriented and is either Haken or geometric, and f is orientation-preserving.

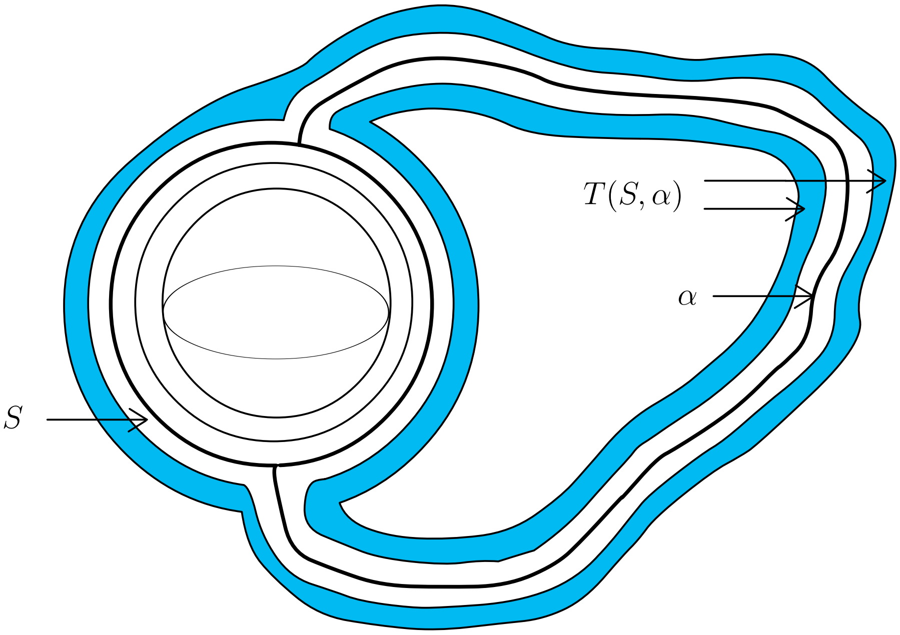
3. What are slides?

A slide homeomophism is a kind of homeomophism $s(S,\alpha)$ on 3-manifold, depending on a shpere S and a path α touching on S.

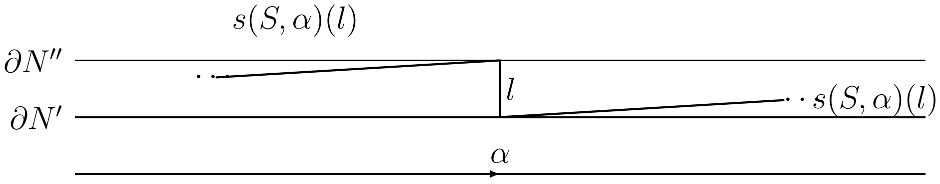
Take two regular neighborhoods N' and N'' ($N' \subset int(N'')$) of $S \cup \alpha(I)$ in M. Then $int(N'' - N')$ have two components which are homemomorphic to $S^2 \times (0,1)$ and $T^2 \times (0,1)$ (written as $T(S,\alpha)$) respectively.

Using a coordinate function $c\colon T(S,\alpha) \to T^2 \times (0,1)$, $s(S,\alpha)\colon M \to M$ is defined by

$$s(x) = \begin{cases} c^{-1}(\theta + 2\pi t, \varphi, t) & \text{if } x = c^{-1}(\theta,\varphi,t) \in T(S,\alpha), \\ x & \text{otherwise.} \end{cases}$$



The behaviour of a slide homeomorphim on its support set is illustrated as following.



A classical result in 3-manifold is:

Proposition 3.1. *Any oriented closed 3-manifold can be written as*

$$M_1 \# M_2 \# \cdots \# M_n \# (\#_k S^2 \times S^1)$$

where M_i is irreducible (no essential S^2) and $k \geq 0$.

In dimension 2, any oriented closed 2-manifold can be written as connect sum of T^2's.

Proposition 3.2. ([8]) *Any orientation-preserving homeomorphism on M^3 is isotopic to a composition of homeomorphisms of the following four types:*

i) summand-preserving,
ii) interchanges of summands,
iii) spins of $S^2 \times S^1$,
iv) silde homeomorphisms.

4. Main Results

Theorem 4.1. *(*[10, Theorem 3.2]*) Let $f = s(S_m, \alpha_m) \circ \cdots \circ s(S_1, \alpha_1)$ be a homeomorphism composed by finitely many slides. Then, up to an isotopy, any component of $M - \cup_{i=1}^{m} T(S_i, \alpha_i)$ is an isolated fixed point set of f with zero fixed point indices.*

Define

$$MI(\alpha_i, S_j) =: \min\{|\alpha \cap S_j| : \alpha \simeq \alpha_j \, rel\{0,1\}, \alpha \text{ has no self intersection}\}.$$

Theorem 4.2. *(*[10, Theorem 4.5]*) Let $f = s_2(S_2, \alpha_2) \circ s_1(S_1, \alpha_1)$ be a homeomorphism composed by two slide homeomorphisms. Let $\{{q^*}_{(1,2;1)}, \ldots, {q^*}_{(1,2;MI(\alpha_1,S_2))}\}$ and $\{{q^*}_{(2,1;1)}, \ldots, {q^*}_{(2,1;MI(\alpha_2,S_1))}\}$ be the set of minimal intersection points of $\alpha_1 \cap S_2$ and $\alpha_2 \cap S_1$ respectively.*

Then, up to an isotopy, f has $2MI(\alpha_1, S_2)MI(\alpha_2, S_1)$ fixed points on $T(S_1, \alpha_1) \cup T(S_2, \alpha_2)$: $x_{(1,2;j,k)}$ and $x_{(2,1;k,j)}$, where $j = 1, 2, \ldots, MI(\alpha_1, S_2)$, $k = 1, 2, \ldots, MI(\alpha_2, S_1)$

Their fixed point coordinates are:

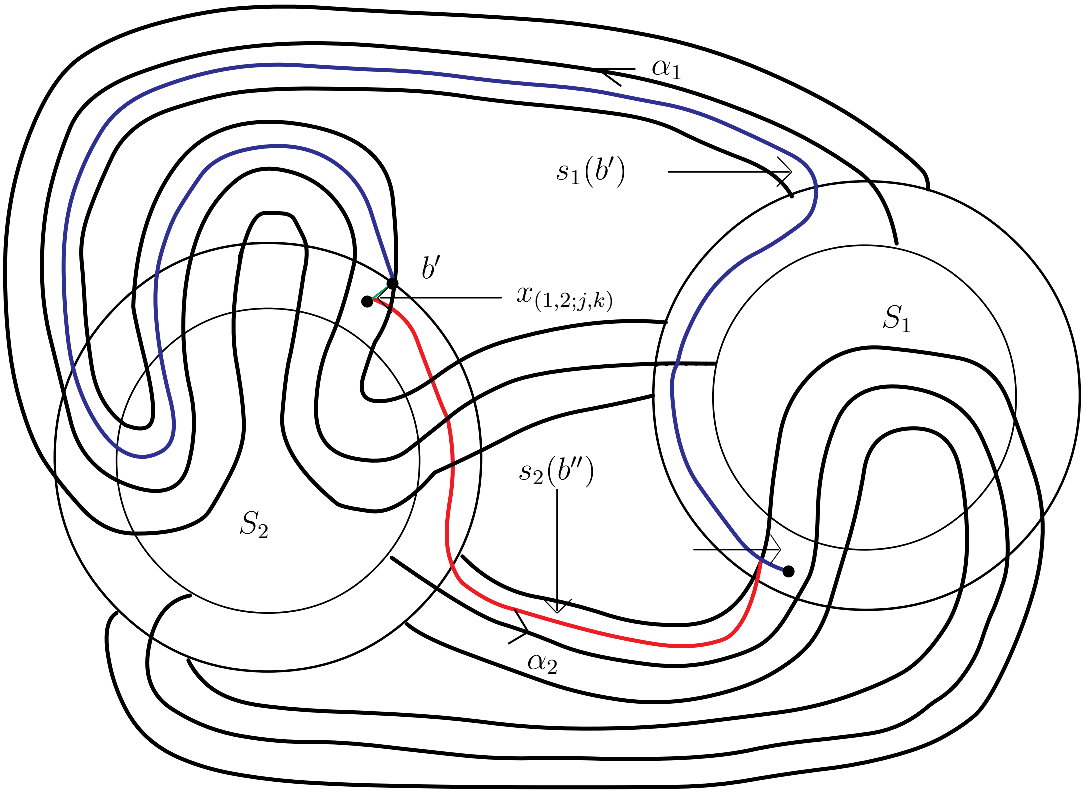
$$cd(f, x_{(1,2;j,k)}) = \langle {\alpha'}_{(2,1;k)} f({\alpha'}_{(1,2;j)}) \rangle, \quad cd(f, x_{(2,1;k,j)}) = \langle ({\alpha'}_{(2,1;k)})^{-1} f({\alpha''}_{(1,2;j)}) \rangle,$$

where $j = 1, 2, \ldots, MI(\alpha_1, S_2)$, $k = 1, 2, \ldots, MI(\alpha_2, S_1)$, and where ${\alpha'}_{(j,k,l)}$ is the sub path of α_j from $\alpha_j(0)$ to ${q^*}_{(j,k,l)}$ and ${\alpha''}_{(j,k,l)}$ is the sub path of α_j from ${q^*}_{(j,k,l)}$ to $\alpha_j(1)$.

Their fixed point indices are:

$$ind(f, x_{(1,2;j,k)}) = {I^*}_{(1,2;j)} {I^*}_{(2,1;k)}, \; ind(f, x_{(2,1;k,j)}) = -{I^*}_{(1,2;j)} {I^*}_{(2,1;k)},$$

where ${I^*}_{(i_1,i_2;i_3)}$ is the algebraic intersection number of α_{i_1} and S_{i_2} at ${q^*}_{(i_1,i_2;i_3)}$. □



As an application of our above results, we given some interesting examples.

Let $M = T_1^3 \# T_2^3 \# T_3^3$. Then $\pi_1(M) = Z^3 * Z^3 * Z^3$ with generators g_{jk}, $j, k = 1, 2, 3$. Take S_j, $j = 1, 2$, to be two different spheres between the summand T_j^3. Consider $f = s(S_2, \alpha_2) \circ s(S_1, \alpha_1)$.

(1) $\langle \alpha_1 \rangle = g_{21}$, $\langle \alpha_2 \rangle = g_{12}$. We have $N(f) = 0$([10, Example 5.1]).

(2) $\langle \alpha_1 \rangle = (g_{31} g_{21})^m$, $\langle \alpha_2 \rangle = g_{32} g_{12} g_{33}$. We have $N(f) = 8m$([10, Example 5.1]).

The Lefschetz numbers of all such kind of homeomorphisms are zero (see [10, Proposition 4.3])!

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