Unknotting torus-covering knots

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Today's talk

F : an ori. surface knot

$u(F) = \min\{n \mid h^1(F; \cup_{i=1}^n h_i) \text{ is unknotted}\}$: the unknotting number

$\mathcal{S}_m(a,b)$: a torus-covering knot, an ori. surface knot determined from commutative m-braids a, b

p: an odd prime, m: a positive integer, Δ: a full twist of m strings

Main Theorem $\exists$ m-braid b s.t. for $\forall n$

$$u(\mathcal{S}_m(b, \Delta^{ln})) = m-1, \text{ where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even.} \end{cases}$$

Theorem A For $\forall$ m-braid b and $\forall l$, $u(\mathcal{S}_m(b, \Delta^l)) \leq m-1$.
(We use charts.)

Theorem B $\exists$ m-braid b s.t. for $\forall n$

$$u(\mathcal{S}_m(b, \Delta^{ln})) \geq m-1, \text{ where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even.} \end{cases}$$

(We use p-colorings and Iwakiri's Theorem.)

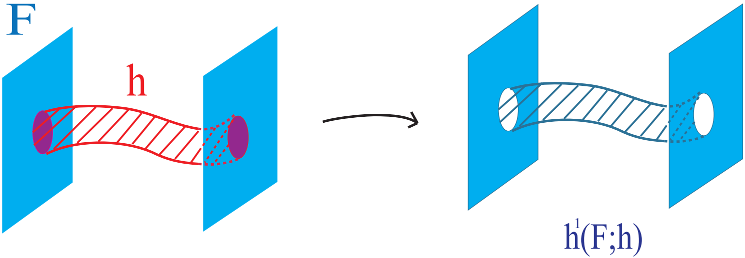
Contents

1. Unknotting numbers

F: an ori. surface knot

i.e.  $\hookrightarrow \mathbb{R}^4$, a smooth embedding

h: 1-handle



a $h^1(F;h)$: 1-handle surgery along h

An unknotted surface knot is a surface knot F

s.t. $F \sim$  $\subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$.

Fact (Hosokawa–Kawauchi, '79)

$\forall F$: an ori. surface knot
$\exists h_1, \ldots, h_n$: disjoint ori. 1-handles
s.t. $h^1(F; h_1 \cup \cdots \cup h_n)$ is an unknotted surface knot.

Definition

F : an oriented surface knot

$$u(F) = \min\{n \mid h^1(F; h_1 \cup \cdots \cup h_n) \text{ is unknotted}\}$$

the unknotting number of F

2. Braided surfaces and their chart desriptions

Definition

Σ: a closed surface
S: a closed surface embedded in $D^2 \times \Sigma$
S is called a braided surface over Σ of degree m
$\stackrel{\text{def.}}{\Longleftrightarrow} p_\Sigma|_S : S \to \Sigma$ is a branched covering map of degree m,
where $p_\Sigma : D^2 \times \Sigma \to \Sigma$ is the projection.

S is called simple if $\#(S \cap p_\Sigma^{-1}(x)) = m - 1$ or m for $\forall x \in \Sigma$.

x_0: a base pt. of Σ.
S, S': braided surfaces over Σ of degree m
S and S' are equivalent if $\exists\ f : D^2 \times \Sigma \to D^2 \times \Sigma$, fiber-preserving ambient isotopy rel $p_\Sigma^{-1}(x_0)$ s.t. $f(S) = S'$.

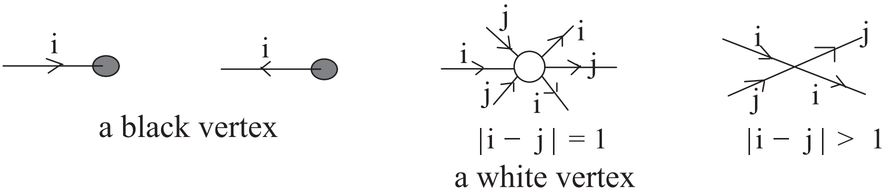
Definition

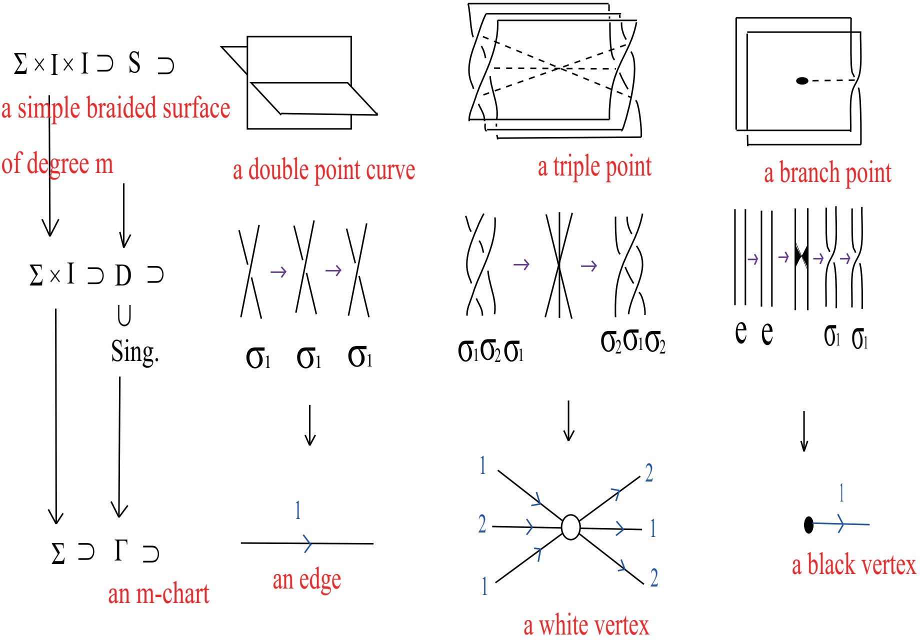
A finite graph Γ on Σ is called an *m-chart* on Σ if it satisfies the following conditions:

(i) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.

(ii) Every vertex has degree 1, 4, or 6.

(iii) The adjacent edges around each vertex are oriented and labeled as shown in the Figure.




Σ × I × I ⊃ S ⊃
a simple braided surface
of degree m
a double point curve
a triple point
a branch point
Σ × I ⊃ D ⊃
∪
Sing.
σ1 σ1 σ1
σ1σ2σ1
σ2σ1σ2
e e σ1 σ1
Σ ⊃ Γ ⊃
an m-chart
1
an edge
1 2 2 1 1 2
a white vertex
1
a black vertex

Fact (Kamada)

Any oriented surface knot is presented by a simple braided surface over S^2.

In other words:
Any oriented surface knot is presented by an m-chart on S^2.

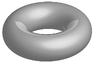
$\{$Simple braided surfaces over $\Sigma\}/\sim$
$\stackrel{1:1}{\longleftrightarrow} \{$ m-charts on $\Sigma\}/$C-move equivalence

Fact

Γ, Γ': *m-charts on* S^2
F, F': *surface knots presented by* Γ, Γ'
If $\Gamma \sim \Gamma'$, *then* $F \sim F'$

3. Torus-covering knots

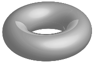
Definition

T: the standard torus in $\mathbb{R}^4$, i.e.  $\subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$
$N(T)$: a tubular neighborhood of T in $\mathbb{R}^4$
F: an oriented surface knot

F is a torus-covering knot
$\overset{\text{def.}}{\Longleftrightarrow}$ F is a simple braided surface over T

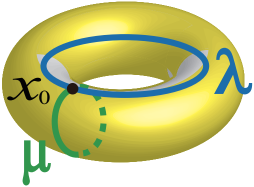
If F has no branch points as a braided surface over T, then F is a T^2-knot.

Definition

T: the standard torus in $\mathbb{R}^4$, i.e.  $\subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$
$N(T)$: a tubular neighborhood of T in $\mathbb{R}^4$
F: an ori. surface knot

F is a torus-covering T^2-knot
$\overset{\text{def.}}{\Longleftrightarrow}$ $p|_F : F \to T$ is an unbranched covering map of degree m

$p : N(T) \to T$: a projection



$$\begin{array}{ccccc} N(T) & \supset & \boxed{F \cap p^{-1}(\mu), F \cap p^{-1}(\lambda)} & \overset{\text{cut at } p^{-1}(x_0)}{\rightsquigarrow} & \text{classical } m\text{-braids} \\ \downarrow p & & \downarrow & & \| \\ T & \supset & \mu, \lambda & & \color{red}{\text{basis braids}} \end{array}$$

Lemma

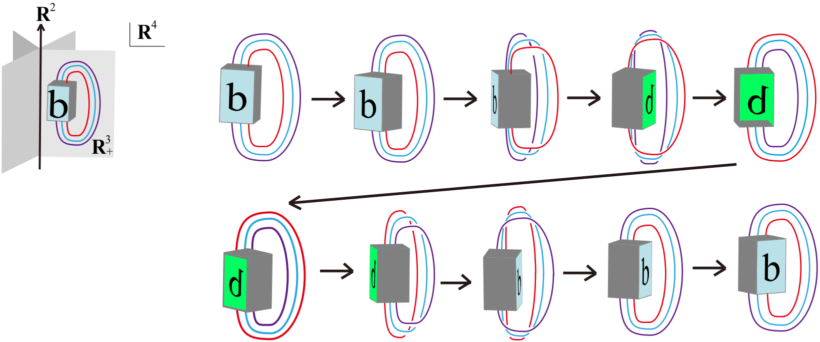
(1) Basis braids are commutative.

(2) For commutative m-braids a and b,

$\boxed{\exists! \text{ a torus-covering } T^2\text{-knot with basis braids } a \text{ and } b}$. $= \mathcal{S}_m(a, b)$

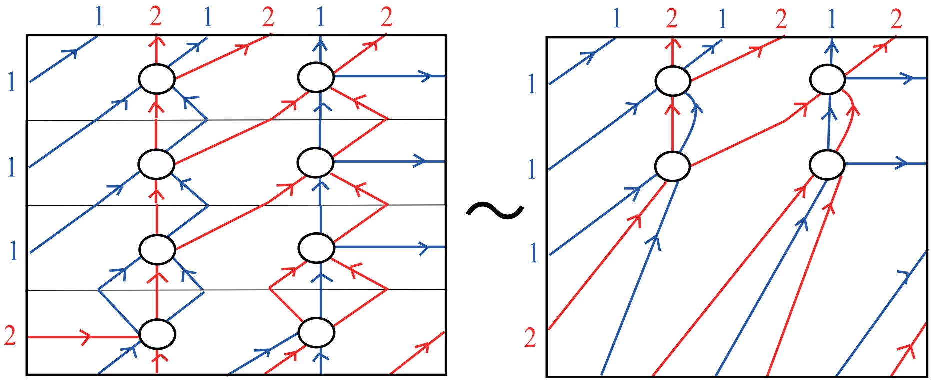
Example ($\mathcal{S}_m(b, \Delta)$)

$\Delta =$  : a full twist



- A torus-covering knot is presented by an m-chart on T.
- Γ, Γ': m-charts on T
  F, F': torus-covering knots presented by Γ, Γ'
  If $\Gamma \sim_{\text{C-move}} \Gamma'$, then $F \sim F'$.

e.g. $\mathcal{S}_3(\sigma_1^3\sigma_2, (\sigma_1\sigma_2)^3)$



4. Main Results

p: an odd prime, m: a positive integer, Δ: a full twist of m strings

Main Theorem

$\exists$ m-braid b s.t. for $\forall n$,

$$u(\mathcal{S}_m(b, \Delta^{ln})) = m - 1, \text{ where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even.} \end{cases}$$

Theorem A

For $\forall$ m-braid b and $\forall l$, $u(\mathcal{S}_m(b, \Delta^{l})) \leq m - 1$.

Theorem B

$\exists$ m-braid b s.t. for $\forall n$,

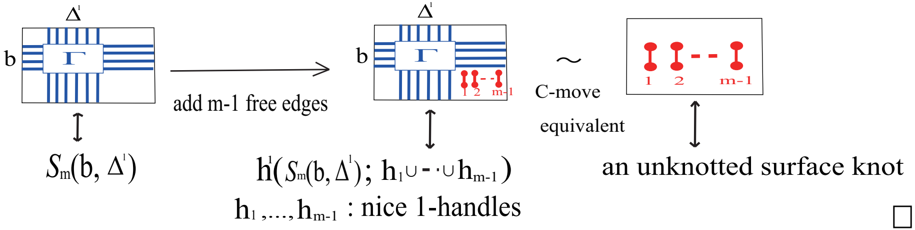
$$u(\mathcal{S}_m(b, \Delta^{ln})) \geq m - 1, \text{ where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even.} \end{cases}$$

5. Upper bound

Theorem A

For ∀ *m-braid* b *and* ∀l, $u(\mathcal{S}_m(b, \Delta^l)) \leq m - 1$.

Outline of the proof.



□

cf. Fact (Kamada)

F*: an ori. surface knot*

Γ*: an m-chart on* S^2 *presenting* F

Then $u(F) \leq w(\Gamma) + m - 1$, *where* $w(\Gamma) = \#\{white\ vertices\}$.

6. Lower bound

Definition

X : a set with $* : X \times X \to X$: a binary operation

X is a quandle

$\overset{\text{def.}}{\Longleftrightarrow}$

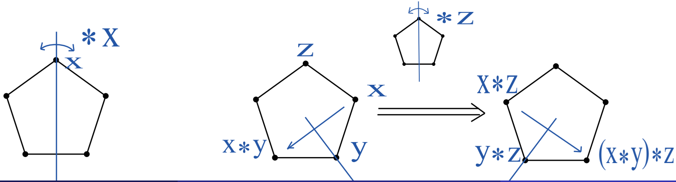
(i) for $\forall x \in X$, $x * x = x$

(ii) for $\forall x \in X$, $*x : X \to X$ is a bijection

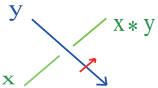
(iii) for $\forall x, \forall y, \forall z \in X$, $(x * y) * z = (x * z) * (y * z)$

Example

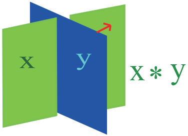
$R_p = \mathbb{Z}/p\mathbb{Z}$, $x * y = 2y - x$: a dihedral quandle



D: a diagram of a classical knot K
A X-coloring C of D is a map
$C : \{\text{arcs of } D\} \to X$ s.t.



D: a diagram of a surface knot F
A X-coloring C of D is a map
$C : \{\text{sheets of } D\} \to X$ s.t.



$\mathrm{Col}_X(D) \overset{def.}{=} \{X\text{-colorings of } D\}$
$|\mathrm{Col}_X(D)|$: invariant of K or F

A R_p-coloring is called a p-coloring.
We denote $|\mathrm{Col}_{R_p}(D)|$ by $|\mathrm{Col}_p(K)|$ or $|\mathrm{Col}_p(F)|$.

Fact (Iwakiri,2006)

F: an ori. surface knot
If $|\mathrm{Col}_p(F)| = p^k$, *then* $u(F) \geq k-1$.

Example (Iwakiri)

F: $2r$-twist spun 2-knot of $7_7, 9_{11}, 9_{15}, 9_{17}$ ($r \neq 0$)
Then $u(\#_n F) = n$,
where $\#_n F$ is a connected sum.

($u(F) = 1$, $|\mathrm{Col}_3(S)| = 3^2$, $|\mathrm{Col}_3(\#_n S)| = 3^n$)

Remark $\tau(F)$: triple pt. canceling number
$a_0(\Phi_\kappa(F)) \rightsquigarrow$ lower bound of $\tau(F)$
(If $|\mathrm{Col}_p(F)| = p^k$ and $|a_0(\Phi_\kappa(F))| < p^{k-l}$, then $\tau(F) > l+1$.)

Theorem B

$\exists$ an m-braid b s.t. for $\forall n$,

$$u(\mathcal{S}_m(b, \Delta^{ln})) \geq m - 1, \text{ where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even} \end{cases}$$

Outline of the proof.

Let b be an m-braid s.t. $\hat{b}$ is a knot and $|Col_p(\hat{b})| = p^m$, e.g. $b = \sigma_1^p \sigma_2^p \cdots \sigma_{m-1}^p$.
Then we can show that $|Col_p(\mathcal{S}_m(b, \Delta^{ln}))| = |Col_p(\hat{b})| = p^m$.
Hence, by Iwakiri's theorem, $u(\mathcal{S}_m(b, \Delta^{ln})) \geq m - 1$. □

Example

(1) For an m-braid $b = \sigma_1^p \sigma_2^p \cdots \sigma_{m-1}^p$ and $\forall n$,

$$u(\mathcal{S}_m(b, \Delta^{ln})) = m - 1, \text{where } l = \begin{cases} 2 & \text{if } m \text{ is odd} \\ p & \text{if } m \text{ is even.} \end{cases}$$

(2) For a 3-braid $b = (\sigma_1 \sigma_2^{-1})^4$ and $\forall n$,

$$u(\mathcal{S}_3(b, \Delta^{2n})) = 2.$$

Question

Is there an m-braid b s.t. $\tau(\mathcal{S}_m(b, \Delta^l)) = m - 1$?
In order to use Iwakiri's Theorem, we must find b s.t.
$a_0(\Phi_f(\mathcal{S}_m(b, \Delta^l)) = p$.

$\tau(F)$: triple pt. canceling number

Remark

For $b = \sigma_1^3\sigma_2^3$ or $(\sigma_1\sigma_2^{-1})^3$, and $\forall n$,

$$\Phi_{\theta_3}(\mathcal{S}_3(b, \Delta^{2n})) = \{0, 0, \ldots, 0\}.$$

Thus, for this case,

$$a_0(\Phi_{\theta_3}(\mathcal{S}_3(b, \Delta^{2n})) = 3^3,$$

and by Iwakiri's Theorem and Main Theorem, we can only say

$$0 \leq \tau(\mathcal{S}_3(b, \Delta^{2n})) \leq u(\mathcal{S}_3(b, \Delta^{2n})) = 2.$$