Milnor's invariants and Hirzebruch-type invariants of homology cylinders

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The 8th East Asian School of Knots and Related Topics

Introduction

- Homology cylinder $\leftsquigarrow$ string link
  $\nwarrow$ mapping class group
  (Homology cobordism group $\mathcal{H}$ $\leftsquigarrow$ string link concordance group)

- $\tilde{\mu}$-invariant of homology cylinders
  ($\leftsquigarrow$ μ-invariant of string links)
  $\leftsquigarrow$ Milnor's $\bar{\mu}$-invariant of links

- Hirzebruch-type invariant on a filtration $\mathcal{H}(q)$
  defined using $\tilde{\mu}$-invariants

Hirzebruch-type invariants

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and given $\phi\colon \pi_1(M) \to \mathbb{Z}_d$ $\qquad (d = p^r)$,

$$\lambda(M,\phi) := [\lambda_W^{\mathbb{Q}(\zeta_d)}] - [\lambda_W^{\mathbb{Z}}] \quad \in L^0(\mathbb{Q}(\zeta_d))$$

$\mathbb{Q}(\zeta_d)$-coefficient intersection form

ordinary intersection form

where W is 4-manifold such that

$$\begin{array}{ccc} M = \partial W & \xrightarrow{\ \phi\ } & BG \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ W & & \end{array}$$

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I defined it on $\widehat{F}$-homology cylinders analogously.

Let's extend the domain!

Hirzebruch-type invariants

For a closed 3-manifold M, iterated p-covers $M_{(h)} \to \cdots \to M_{(1)} \to M$
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Homology cylinders

Let $\Sigma = \Sigma_{g,n}$ be an oriented surface with genus g and n boundary components.

Definition [Goussarov(1999), Habiro(2000)]

A homology cylinder over Σ consists of a compact oriented 3-manifold M with two embeddings $i_+,\ i_-\colon \Sigma \hookrightarrow \partial M$ such that:

1. i_+ is orientation-preserving and i_- is orientation-reversing
2. $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$
3. $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$
4. $i_+, i_-\colon H_*(\Sigma;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ are isomorphisms.

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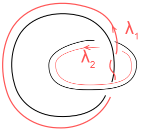

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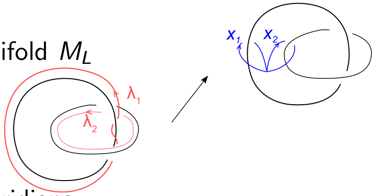

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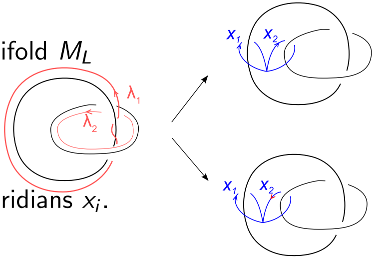

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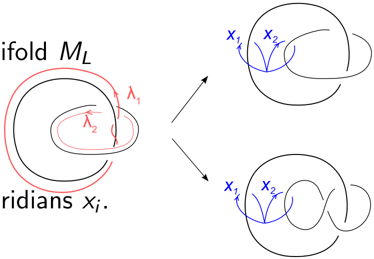

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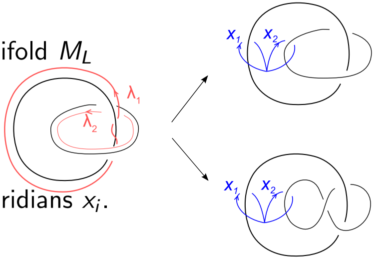

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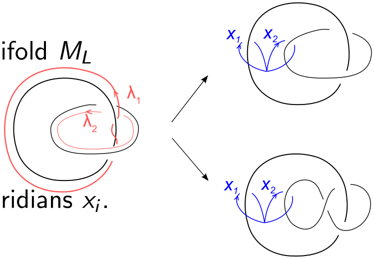

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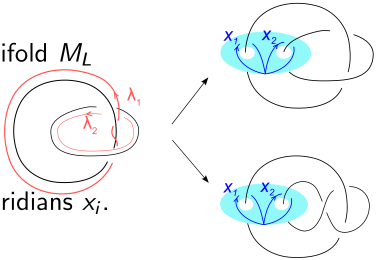

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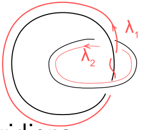

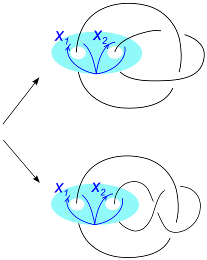

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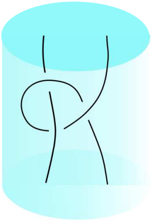
3. link concordance group

string links and μ-invariants

k-string link $sL = \text{Image}(\{1, \cdots, k\} \times I \hookrightarrow D^2 \times I)$

where $(i, t) \mapsto (p_i, t)$ for $i = 1, \cdots, k$ and $t = 0, 1$.

(p_i's are fixed distinct points in $\text{int}(D^2)$)



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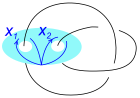

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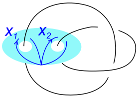

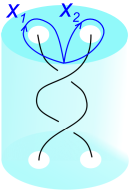

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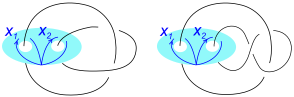

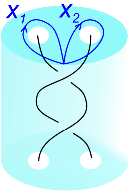

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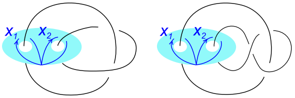

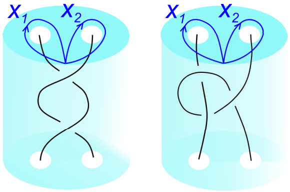

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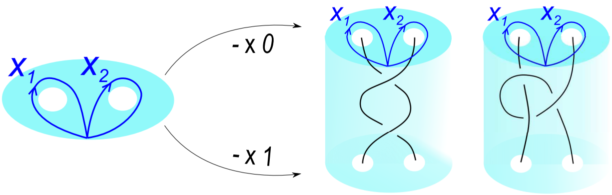

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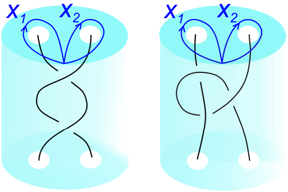

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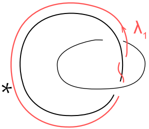

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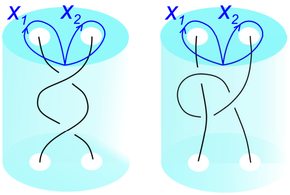

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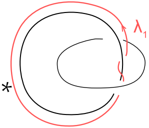

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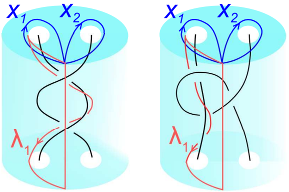

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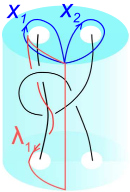

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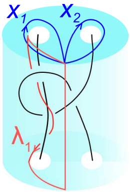
0. $H_*(D^2 - \{p_i\}) \xrightarrow[-\times 1]{-\times 0} H_*(D^2 \times I - sL)$ are isomorphisms for all $*$
1. closed manifold : the zero surgery manifold M_L of $L = \widehat{sL}$
   $\pi_1(M_L) = \pi_1(D^2 \times I - sL)/\langle\langle \lambda_i \rangle\rangle$
   where λ_i's are "longitudes" of sL.
2. μ-invariants [Habegger-Lin]:

$$\begin{array}{ccccc} F & \xrightarrow{i_+} & \pi_1(D^2 \times I - sL) & & \ni \lambda_i \\ \downarrow & & \downarrow & & \uparrow \\ F/F_q & \xrightarrow{\cong} & \pi_1(D^2 \times I - sL)/\pi_1(D^2 \times I - sL)_q & & | \\ \mu_i & \leftarrow & \lambda_i & & \swarrow \end{array}$$



3. string link concordance group

Generalization

Consider the followings as generalization:

1. string links in a homology-$(D^2 \times I)$
   (whose boundary is identified with $\partial(D^2 \times I)$)
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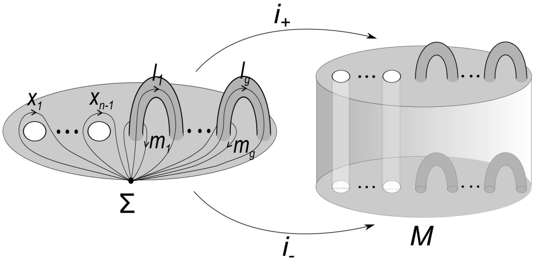
String links in homology-$(\Sigma_{g,1} \times I)$
whose boundary is $\partial(\Sigma_{g,1} \times I)$ with endpoint condition
$=$ Homology cylinders over $\Sigma_{g,n}$

homology cylinders and $\tilde{\mu}$-invariants

homology cylinder (M, i_+, i_-) over $\Sigma_{g,n}$ = compact oriented 3-manifold M

$i_+, i_-\colon \Sigma \hookrightarrow \partial M(\simeq \partial(\Sigma \times I))$

such that i_+, i_- induces $H_*(\Sigma) \xrightarrow{\cong} H_*(M)$.



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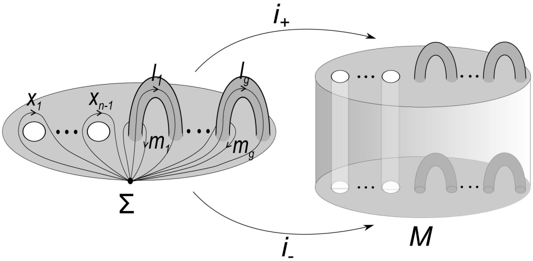

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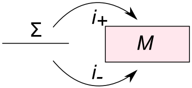

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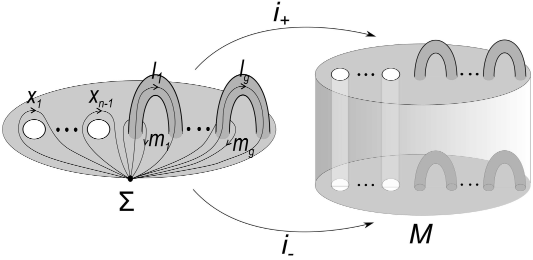

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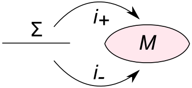

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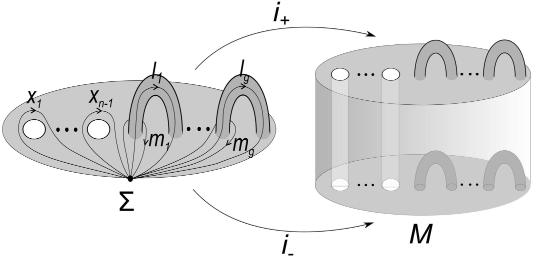

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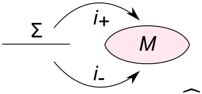

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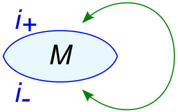

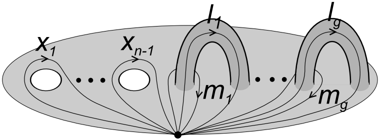

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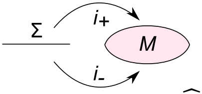

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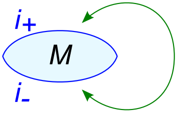

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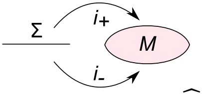

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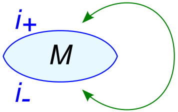

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3. homology cobordism group $\mathcal{H}_{g,n}$

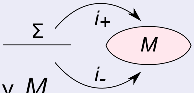
Homology cylinders

Let $\Sigma = \Sigma_{g,n}$ be an oriented surface with genus g and n boundary components.

Definition [Goussarov(1999), Habiro(2000)]

A homology cylinder over Σ consists of a compact oriented 3-manifold M with two embeddings $i_+,\ i_-\colon \Sigma \hookrightarrow \partial M$ such that:

1. i_+ is orientation-preserving and i_- is orientation-reversing
2. $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$
3. $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$
4. $i_+, i_-\colon H_*(\Sigma;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ are isomorphisms.



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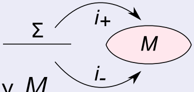

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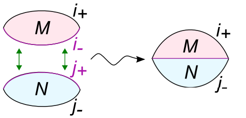
$\mathcal{C}_{g,n} :=$ {homology cylinders} / (isomorphism)
with $M \cdot N := (M \cup_{i_-=j_+} N, i_+, j_-)$ $\rightsquigarrow$ a monoid



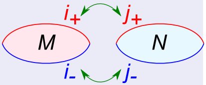
Homology cobordism

Let (M, i_+, i_-) and (N, j_+, j_-) be homology cylinders over Σ.

Definition [Levine(2001), Garoufalidis-Levine(2005)]

M and N are said to be homology cobordant
if $\exists$ a compact oriented 4-manifold W such that:

1. $\partial W = \widehat{M \cdot -N} = M \cup (-N)/(i_\pm(x) = j_\pm(x),\ x \in \Sigma)$
2. $M \hookrightarrow W$, $N \hookrightarrow W$ induce homology isomorphisms.



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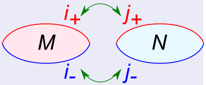

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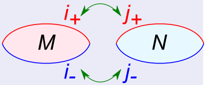

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Examples

- $\mathcal{H}_{0,0} = \mathcal{H}_{0,1}$ = homology cobordism group of homology-S^3
- $\mathcal{H}_{0,2}$ = knot concordance group (in homology-S^3)
- $\mathcal{H}_{g,n} = (n-1)$-string link concordance group in homology-$(\Sigma_{g,1} \times I)$
- mapping class group over $\Sigma_{g,n} \hookrightarrow \mathcal{H}_{g,n}$

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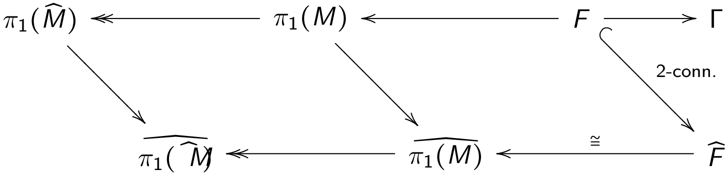

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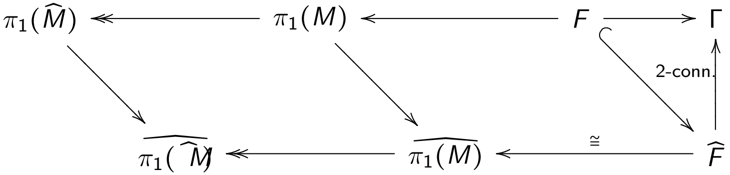

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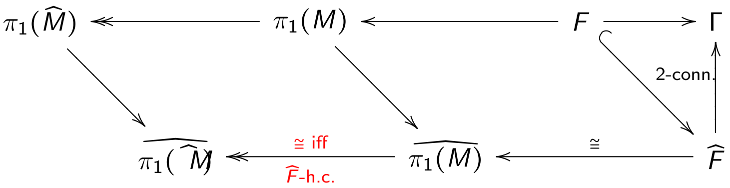

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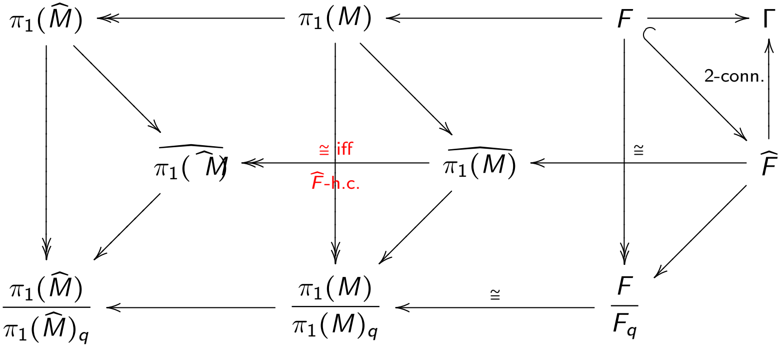

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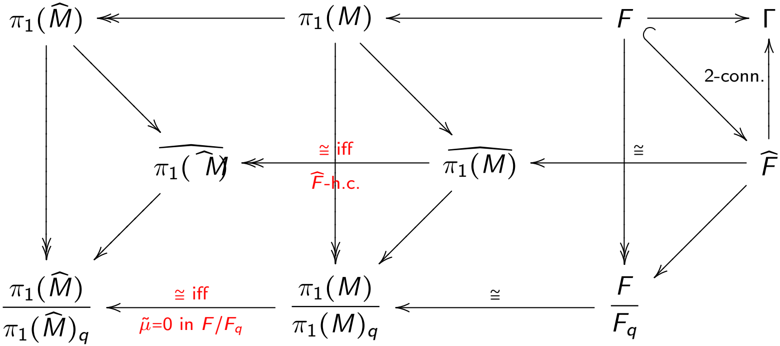

A Hirzebruch-type invariants on $\mathcal{H}(q)$

Define $\mathcal{C}(q) := \{M \in \mathcal{C}_{g,n} \mid \tilde{\mu}(M) = 0 \text{ in } F/F_q\}$
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Given a $\mathbb{Z}_d$-valued p-structure $\mathcal{T}$ of height h for F/F_q ($d = p^r$)

$$\mathcal{T} = (\{\frac{F_{(h)}}{F_q} \triangleleft \cdots \triangleleft \frac{F_{(1)}}{F_q} \triangleleft \frac{F_{(0)}}{F_q} = \frac{F}{F_q}\}, \ \phi\colon \frac{F_{(h)}}{F_q} \to \mathbb{Z}_d)$$

(equivalently, iterated p-covers of Σ with $\phi\colon \pi_1(\Sigma_{(h)}) \to \mathbb{Z}_d$ under F_q)

$$\lambda_{\mathcal{T}}(M) \quad := \quad \lambda(\widehat{M}_{(h)}, \phi) \in L^0(\mathbb{Q}(\zeta_d))$$

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Theorem

1. $\lambda_{\mathcal{T}}$ is well-defined in $\mathcal{C}(q)$.
2. $\lambda_{\mathcal{T}}$ is well-defined in $\mathcal{H}(q)$, i.e. a homology cobordism invariant.

Q. When $\lambda_{\mathcal{T}}$ has additivity?

Conditions for additivity of $\lambda_{\mathcal{T}}$

Let $F_{(h+1)} := \mathrm{Ker}\{\phi\colon F_{(h)} \to \mathbb{Z}_d\}$.

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Corollary

If $\mathcal{T}$ satisfies $F_q \subset [F_{(h+1)} : F_{(h+1)}]$,
then $\lambda_{\mathcal{T}}$ is a homomorphism on the whole $\mathcal{H}(q)$.

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For any p-structure of F, $\lambda_{\mathcal{T}}$ is a homomorphism on $\cap_q \mathcal{H}(q)$.

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Lemma

For a finitely generated group G
and a p-tower $G_{(h)} \triangleleft \cdots \triangleleft G_{(1)} \triangleleft G_{(0)} = G$,
$G_{(h)}$ contains G_q for some q.

Proof. Use p-mixed commutator series.

To do

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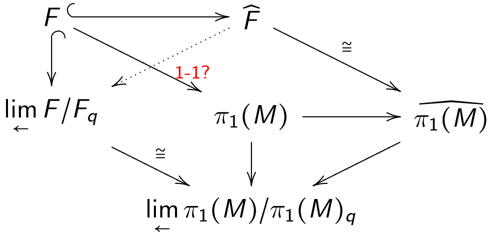
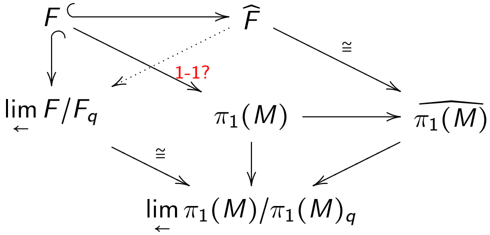


homology cylinder with $\tilde{\mu} = 0 \Leftarrow \widehat{F}$-homology cylinder
$\Rightarrow ?$

To do

Q. Find $\mathcal{T}$ satisfying $F_q \subset [F_{(h+1)} : F_{(h+1)}]$,
so that $\lambda_{\mathcal{T}}$ is a homomorphism on $\mathcal{H}(q)$

Q. $\cap_q \mathcal{H}(q)$ vs. $\{\widehat{F}\text{-homology cylinders}\}/\sim$?
i.e. homology cylinder with all $\tilde{\mu} = 0$ vs. $\widehat{F}$-homology cylinders



homology cylinder with $\tilde{\mu} = 0 \Leftarrow \widehat{F}$-homology cylinder
$\Rightarrow$? Yes if $\swarrow$ is 1-1.

To do

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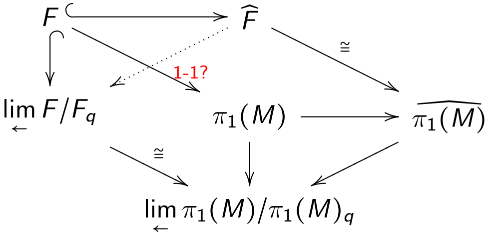

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cf. Conjecture: link with $\bar{\mu} = 0 \Rightarrow \widehat{F}$-link ?

Thank you~ ★