Coloring link diagrams by Alexander quandles

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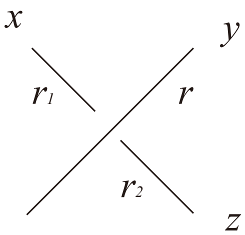
A *coloring* on an oriented classical knot diagram D by a set X is a function $C : R \to X$, where R is the set of over-arcs in the diagram.



Definition (D. Joyce and S. Matveev, 1982)

A *quandle* is a non-empty set X equipped with a binary operation $*$ satisfying the following three axioms:

(Q1) For any $x \in X$, $x * x = x$.

(Q2) For any $x, y \in X$, there is a unique element $z \in X$ such that $x = z * y$.

(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The property (Q2) is equivalent to the following property that

(Q2′) There is a binary operation $\bar{*} : X \times X \to X$ such that for any $x, y \in X$, $(x * y)\bar{*}y = (x\bar{*}y) * y = x$.

A *quandle homomorphism* is a map between two quandles preserving the quandle operation.

Example

- Let X be a subset of a group closed under conjugations. Then X is a quandle, called a *conjugation quandle*, under the operation $x * y = y^{-1}xy$ for all $x, y \in X$.
- Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring over the integers. Then any Λ-module M has a quandle structure, called an *Alexander quandle*, under the operation

$$a * b = ta + (1 - t)b$$

for $a, b \in M$.

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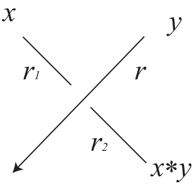

Remark

- *Locally the colors do not depend on the orientation of the under-arc.*
- *Any constant function $C_q : R \to Q$ at $q \in Q$ is a coloring, called the trivial coloring at $q \in Q$.*

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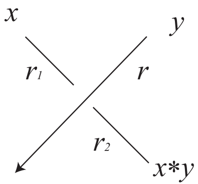

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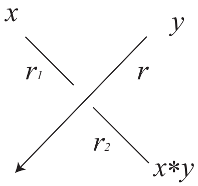

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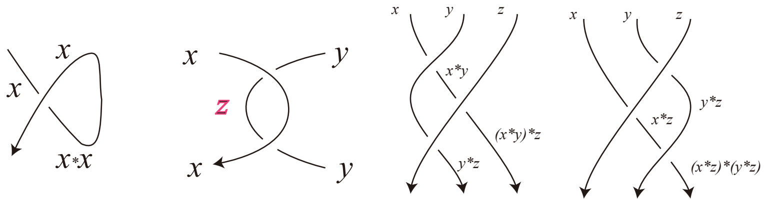
A diagram D of a link L is said to be *colorable* by a quandle Q if it admits a non-trivial coloring by Q.

Proposition

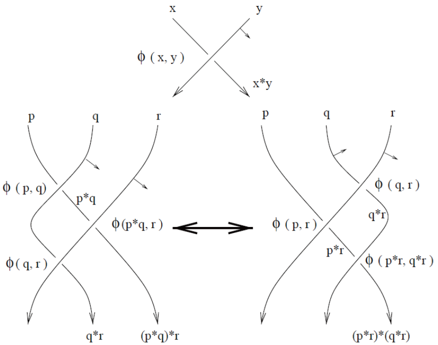
Colorability by a quandle is an invariant of links.

Proof.

The defining Axioms $(Q1), (Q2)$ and $(Q3)$ of the quandle correspond to the Reidemeister Moves Ω_1, Ω_2 and Ω_3, respectively.



Let D be a diagram of a link L colored by a quandle $(X, *)$, and A an abelian group. Let $\phi : X \times X \to A$ be a map. For each crossing of D, assign $\phi(p, q)$ as follows.



Suppose that $\phi(p, r) \cdot \phi(p * r, q * r) = \phi(p, q) \cdot \phi(p * q, r)$. Then $\prod_{\tau\,:\,\text{crossing}} \phi(p, q)^{\varepsilon(\tau)}$ is a link invariant, which depends on the coloring by the quandle Q.

For a map $\phi : X \times X \to A$ satisfying
$\phi(p, r) \cdot \phi(p * r, q * r) = \phi(p, q) \cdot \phi(p * q, r)$,

$$\Phi_Q(L) = \sum_{\text{all colorings by } Q} \prod_{\text{all crossings}} \phi(p, q)$$

is a link invariant, called a *quandle cocycle* invariant.

To get such an invariant, we need to know that

- Given a quandle Q, determine whether a link L is colorable by Q or not.
- How many colorings by Q are there, if exists?
- How can one find such a function $\phi : X \times X \to A$?

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- How many colorings by Q are there, if exists?
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It is known that, for a finite quandle X, there is a one-to-one correspondence between quandle homomorphisms $Q(K) \to X$ and colorings $C : R \to X$.

Proposition (A. Inoue, 2001)

Let p be a prime number, J an ideal of the ring Λ_p and $Q(K)$ a knot quandle. For each $i \geq 0$, we put $e_i(t) = \Delta_K^{(i)}(t)/\Delta_K^{(i+1)}(t)$. Then the number of all quandle homomorphisms of the knot quandle $Q(K)$ to the Alexander quandle Λ_p/J is equal to the cardinality of the module $\Lambda_p/J \oplus \oplus_{i=0}^{n-2}\{\mathbb{Z}_p[t, t^{-1}]/(e_i(t), J)\}$.

Remark

As a corollary, he showed that for an Alexander quandle Λ_p/J and for a knot diagram K, there are only trivial colorings of a diagram of L by the quandle Λ_p/J if and only if the ideal generated by J and $\Delta_K(t)$ is equal to Λ_p, where p is a prime number.

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Linear Algebra with coefficient ring Λ and $\Lambda/(f(t))$

We need to check the properties of matrices whose entries are in Λ or $\Lambda/(f(t))$, where $f(t)$ is a fixed non-zero polynomial in Λ. Consider

$$\begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (*)$$

If the coefficients are in a field, then the solution of the system $(*)$ of linear equations does not changed by the following elementary row operations, and every matrix can be changed to a row-echelon form by using elementary row operations.

- $\mathcal{R}_1$ Row switching(a row within the matrix is switched with another row),
- $\mathcal{R}_2$ Row multiplication(each element in a row is multiplied by a non-zero constant) and
- $\mathcal{R}_3$ Row addition(a row is replaced by the sum of that row and a multiple of another row).

For the matrices with coefficients in Λ or $\Lambda/(f(t))$, we modify the operation $\mathcal{R}_2$ as

- $\mathcal{R}_2'$ Row multiplication for Λ(each element in a row is multiplied by a non-zero polynomial $\alpha(t)$)
- $\mathcal{R}_2'$ Row multiplication for $\Lambda/(f(t))$(each element in a row is multiplied by a non-zero polynomial $\alpha(t)$ with $(\alpha(t), f(t)) = 1$)

Elementary row operations $\mathcal{R}_1, \mathcal{R}_2'$ and $\mathcal{R}_3$ do not change the solution of the system $(*)$, and every matrix with coefficients in Λ or $\Lambda/(f(t))$ can be changed by a row-echelon form by using elementary row operations $\mathcal{R}_1, \mathcal{R}_2'$ and $\mathcal{R}_3$.
We will use the terminology "rank" for the matrices with coefficients in Λ or $\Lambda/(f(t))$, too.

Consider the system $(*')$ of linear equations:

$$A\mathbf{x} = \begin{pmatrix} a_1(t) & * & * & \cdots & * & \cdots & * \\ 0 & 0 & a_2(t) & \cdots & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_k(t) & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (*')$$

Proposition

(1) *If rank*$(A) < n$, $(*')$ *has infinitely many solutions.*

(2) *If rank*$(A) = n$ *and if the coefficient ring is* Λ, $(*')$ *has the only trivial solution.*

(3) *If rank*$(A) = n$ *and if the coefficient ring is* $\Lambda/(f(t))$, *then*

   (i) *if* $a_i(t)$ *and* $f(t)$ *are relatively prime for all* i, *then* $(*')$ *has the only trivial solution;*

   (ii) *if there exist* i *such that* $a_i(t)$ *and* $f(t)$ *are not relatively prime, then* $(*')$ *has infinitely solutions.*

Proposition

For $A = (a_{ij})$ an $n \times n$-matrix with coefficients in Λ, if $a_{11} \neq 0$, then define

$$b_{ij}(t) = \det \begin{pmatrix} a_{11}(t) & a_{1j}(t) \\ a_{j1}(t) & a_{ij}(t) \end{pmatrix}$$

for i, j with $2 \leq i \leq m$ and $2 \leq j \leq n$.
Let $B = (b_{ij})_{(n-1)\times(n-1)}$. Then

$$\det(B) = a_{11}^{n-2} \det(A).$$

Colorability by Alexander quandles

- Q an Alexander quandle
- $\phi : B_n \to GL(n, \Lambda)$ the Burau representation for the braid group B_n, which is defined by

$$\phi(\sigma_i) = \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & t & 1-t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n-i-2} \end{pmatrix}, i = 1, \cdots, n-1.$$

- For a braid $w \in B_n$, suppose that the diagram of w, presented by the standard generators, is colored by Q.
- $(c_1, c_2, \cdots, c_n)$ in Q^n the vector obtained by colors assigned to the top arcs of w in the given order, the *coloring of the braid* $w \in B_n$.
- $(d_1, d_2, \cdots, d_n)$ in Q^n the vector obtained by colors assigned to the top arcs of w in the given order
- $\phi(w)(c_1, c_2, \cdots, c_n)^T = (d_1, d_2, \cdots, d_n)^T$.

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- The coloring $(c_1, c_2, \cdots, c_n)$ of w induces a coloring of the closure $\overline{w}$ if and only if $(c_1, c_2, \cdots, c_n) = (d_1, d_2, \cdots, d_n)$.
- For $A \in M(n, \Lambda)$ an $n \times n$-matrix, put $E(A) = \{(c_1, c_2, \cdots, c_n) \in Q^n | \; A(c_1, c_2, \cdots, c_n)^T = (c_1, c_2, \cdots, c_n)^T\}$.
- For any $w \in B_n$, $E(\phi(w)) \neq \emptyset$ because $\phi(w)(c, c, \cdots, c) = (c, c, \cdots, c)$ for all $c \in Q$.
- Since $\phi(w) \in GL(n, \Lambda) \subset M(n, \Lambda)$ and Q a Λ-module, $\phi(w)$ can be seen as a module homomorphism $\phi(w) : Q^n \to Q^n$ defined by the matrix multiplication $\phi(w)(x) = \phi(w)x$ for all $x \in Q^n$. Since $E(\phi(w))$ is the kernel of $(\phi(w) - id)$, it is a submodule of Q^n.

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- Let D be a diagram of a link L and Q an Alexander quandle.
- Let $\mathcal{VC}_D(Q)$ be the set of all colorings on a link diagram D by Q.
- Define the addition of colorings and scalar multiplication by adding the quandle elements assigned on each arc of D by $\mathcal{C}_1$ and $\mathcal{C}_2$ and by multiplying $\alpha(t)$ to the quandle elements assigned on each arc of D by $\mathcal{C}_1$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two colorings on a diagram D and $\alpha(t)$ is in Λ.
- $\mathcal{VC}_D(Q)$ is a Λ-module under the addition and the scalar multiplication, and is isomorphic to the submodule $E(\phi(w))$ of $\phi(w)$ of Q^n, where D is the diagram of the closure $\overline{w}$ given by the braid diagram.
- A. Inoue mentioned about the sum of colorings and scalar multiplication to colorings by the Alexander quandle Λ_p/J.

Lemma

Let $\phi : B_n \to GL(n, \Lambda)$ be the Burau representation and Q any non-trivial Alexander quandle. If the closure of a braid $w \in B_n$ admits only the trivial coloring by Q, then $rank(\phi(w) - id) = n - 1$. In particular, if Q is torsion-free and finitely generated as a Λ-module, the converse holds.

It is known that, for a braid word $w \in B_n$,

$$\phi(w) = C \begin{bmatrix} \widetilde{\phi}(w) & * \\ 0 & 1 \end{bmatrix} C^{-1} \text{ for some } \; C \in GL(n, \Lambda), \text{ and}$$

$$\det(\widetilde{\phi}(w) - id) = (1 + t + \cdots + t^{n-1})\Delta_L(t),$$

where L is the closure of w and $\Delta_L(t)$ is the reduced Alexander polynomial of the link L.

Lemma

Let $\phi : B_n \to GL(n, \Lambda)$ be the Burau representation and Q any non-trivial Alexander quandle. If the closure of a braid $w \in B_n$ admits only the trivial coloring by Q, then $\mathrm{rank}(\phi(w) - id) = n - 1$. In particular, if Q is torsion-free and finitely generated as a Λ-module, the converse holds.

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where L is the closure of w and $\Delta_L(t)$ is the reduced Alexander polynomial of the link L.

Theorem

If the reduced Alexander polynomial $\Delta_L(t)$ of a link L is zero, then L admits a non-trivial coloring by any non-trivial Alexander quandle Q.

Proof.

Suppose that L is the closure of $w \in B_n$. Since

$\phi(w) - id = C \begin{bmatrix} \widetilde{\phi}(w) - id & * \\ 0 & 0 \end{bmatrix} C^{-1}$ and

$\det(\widetilde{\phi}(w) - id) = (1 + t + \cdots + t^{n-1})\Delta_L(t)$, $\Delta_L(t) = 0$ if and only if $\det(\widetilde{\phi}(w) - id) = 0$ if and only if $\text{rank}(\phi(w) - id) = \text{rank}(\widetilde{\phi}(w) - id) \leq n - 2$. By Lemma 4, L admits a non-trivial coloring by any Alexander quandle Q. □

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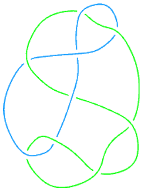
Example

L9n27 is the closure of $w = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 \in B_4$, and its Alexander polynomial is vanishing.



By the matrix calculation, one can see that $(\phi(w) - id)$ can be changed to the following by the elementary row operations $\mathcal{R}_1, \mathcal{R}_2'$ and $\mathcal{R}_3$.

$$\begin{bmatrix} -\frac{-1+4t-3t^2+t^3}{t} & -1+4t-3t^2+t^3 & -\frac{2-6t+7t^2-4t^3+t^4}{t} & -\frac{-1+t}{t} \\ 0 & 0 & \frac{(-1+t)^3}{t^2} & -\frac{(-1+t)^3}{t^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since rank$(\phi(w) - id) = 2$, $(\phi(w) - id)(x_1, x_2, x_3, x_4)^T = (0, 0, 0, 0)^T$ has non-constant solutions, e.g., $(t, 1, 0, 0)$ and $(1 - 2t, -1, 1, 1)$. For any Alexander quandle Q and $q \in Q$, by coloring the top strands of the braid w by $(1 - 2t)q, -q, q$ and q in the given order, one can obtain a non-trivial coloring of $L9n27$ by Q whenever $(1 - 2t)q, -q, q$ are not all equal in Q.

It is well-known that if L is a split link, then $\Delta_L(t) = 0$, but the converse does not hold. There are 11 prime links with up to 11 crossings whose multi-variable Alexander polynomial is 0: $L9n27$, $L10n32$, $L10n36$, $L10n107$, $L11n244$, $L11n247$, $L11n334$, $L11n381$, $L11n396$, $L11n404$ and $L11n406$ in Thistlethwaite Link Table. One can see that $\dim E(\phi(w)) = 2$ for the above 11 links, where w is the braid presentation of the link given in Thistlethwaite Link Table.

Now, assume that the reduced Alexander polynomial $\Delta_L(t)$ of L is non-vanishing. If $\Delta_L(t) = 1$, L admits only the trivial coloring by the quandle $\Lambda/(\Delta_L(t))$ because $\Lambda/(\Delta_L(t)) = \{1\}$.

Theorem

Let L be a link with non-trivial and non-vanishing reduced Alexander polynomial $\Delta_L(t)$. Then L admits a non-trivial coloring by the Alexander quandle $\Lambda/(\Delta_L(t))$.

Proof.

Let $w \in B_n$ be a braid presentation of L. Observe that the sum of entries in each row of the matrix $(\phi(w) - id)$ is zero. Since $\Delta_L(t) \neq 0$, rank$(\phi(w) - id) = n - 1$, so that $(\phi(w) - id)$ can be changed to the matrix of the following form.

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1(n-1)}(t) & a_{1n}(t) \\ 0 & a_{22}(t) & a_{23}(t) & \cdots & a_{2(n-1)}(t) & a_{2n}(t) \\ 0 & 0 & a_{33}(t) & \cdots & a_{3(n-1)}(t) & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)(n-1)}(t) & a_{(n-1)n}(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

□

Since the sum of each row entries is zero, it is enough to solve the following system:

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1(n-1)}(t) \\ 0 & a_{22}(t) & a_{23}(t) & \cdots & a_{2(n-1)}(t) \\ 0 & 0 & a_{33}(t) & \cdots & a_{3(n-1)}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)(n-1)}(t) \end{bmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{1}$$

Since

$$\begin{aligned} & a_{11}(t)a_{22}(t)\cdots a_{(n-1)(n-1)}(t) \\ & \quad = a_{11}(t)^{n-2}a_{22}(t)^{n-3}\cdots a_{(n-2)(n-2)}(t)^{1}\det(\widetilde{\phi}(w) - id). \end{aligned}$$

and since $\det(\widetilde{\phi}(w) - id) = \Delta_L(t)(1 + t + \cdots + t^{n-1})$, there exists j such that $a_{jj}(t)$ is not relatively prime with $\Delta_L(t)$. If $a_{jj}(t)$ is a multiple of $\Delta_L(t)$, it is zero in $\Lambda/(\Delta_L(t))$ so that the rank of the coefficient matrix is less than $n-1$ so that the system (1) has infinitely many solutions.

If all diagonal entries which are not relatively prime with $\Delta_L(t)$ are not a multiple of $\Delta_L(t)$, then the rank of the coefficient matrix is $n-1$. Hence, the system (1) has infinitely many solutions.

From the proof of the above theorem, one can see that

(1) if $f(t)$ is a factor of $\Delta_L(t)$, then L admits a non-trivial coloring by the Alexander quandle $\Lambda/(f(t))$.

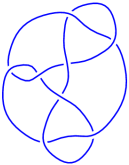
(2) in particular, if $f(t)$ is irreducible, then only one diagonal entry will be zero in $\Lambda/(f(t))$ so that $\mathrm{rank}(\phi(w) - id) = n - 2$ in $\Lambda/(f(t))$.

(3) if $\mathrm{rank}(\phi(w) - id) = n - 2$ in $\Lambda/(f(t))$ and if $(x_1(t), \cdots, x_n(t))$ is a non-trivial coloring of $\overline{w}$ by $\Lambda/(f(t))$, all colorings of $\overline{w}$ by $\Lambda/(f(t))$ are linear combinations of $(x_1(t), \cdots, x_n(t))$ and $(1, \cdots, 1)$.

Example

The knot 8_{15} is the closure of $w = \sigma_1^2\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^3\sigma_3$ in B_4 and its Alexander polynomial is
$\Delta_{8_{15}}(t) = 3 - 8t + 11t^2 - 8t^3 + 3t^4 = (3t^2 - 5t + 3)(1 - t + t^2)$.
Notice that $(\phi(w) - id)$ can be changed to the following matrix by elementary row operations $\mathcal{R}_1, \mathcal{R}_2', \mathcal{R}_3$.

$$\begin{bmatrix} -2+2t-t^2 & -\left(-2t^2-1+3t+t^3\right)(-1+t) & 1-t & 6t^2+2-5t-3t^3+t^4 \\ 0 & \left(-t+1+t^2\right)\left(t^2-3t+3\right) & t-1-t^2 & -6t^2+5t-2+4t^3-t^4 \\ 0 & 0 & 3t^2-5t+3 & 5t-3t^2-3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



Note that every diagonal entries of the above matrix can not be zero in $\Lambda/(\Delta_{8_{15}}(t))$ so that $\text{rank}(\phi(w) - id) = n - 1$. Since the second and the third diagonal entries are not relatively prime with $\Delta_{8_{15}}(t)$, 8_{15} admits infinitely many non-trivial colorings by $\Lambda/(\Delta_{8_{15}}(t))$. For example, the colorings $(5t - 3t^2 - 3, (-1 + t)(3t^2 - 5t + 3), 0, 0)$ and $(t(5t - 3t^2 - 3), 5t - 3t^2 - 3, 0, 0)$ of w give non-trivial colorings of 8_{15}.

For the irreducible factor $(1 - t + t^2)$ of $\Delta_{8_{15}}(t)$, only the second diagonal entry is zero in $\Lambda/(1 - t + t^2)$ so that $\text{rank}(\phi(w) - id) = n - 2$. Note that $(1, 1 - t, 0, 0)$ is a non-trivial coloring of 8_{15} by $\Lambda/(1 - t + t^2)$, and that all colorings of 8_{15} by $\Lambda/(1 - t + t^2)$ are linear combinations of $(1, 1 - t, 0, 0)$ and $(1, 1, 1, 1)$.
For the other irreducible factor $(3t^2 - 5t + 3)$ of $\Delta_{8_{15}}(t)$, one can obtain the similar result.

Example

(1) For the trefoil knot $3_1 = \overline{\sigma_i^3}$, $\Delta_{3_1}(t) = 1 - t + t^2$ is irreducible, and hence rank$(\phi(w) - id) = 0$. Hence one can choose any non-constant element, say $(1, 0)$, in $(\Lambda/(\Delta_{3_1}(t)))^2$ as a non-trivial coloring of 3_1 (the first arc is colored by 1 and the second by 0). Since $\{(1, 0), (1, 1)\}$ generates $(\Lambda/(\Delta_{3_1}(t)))^2$, every element of $(\Lambda/(1 - t + t^2))^2$ can be a coloring of 3_1.

(2) The knot 8_{20} is the closure of $w = \sigma_1^3\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-1}$ in B_4 and $\Delta_{8_{20}}(t)$ $= 1 - 2t + 3t^2 - 2t^3 + t^4 = (1 - t + t^2)^2$. One can see that $(\phi(w) - id)$ can be changed to the following row-echelon form by elementary row operations $\mathcal{R}_1, \mathcal{R}_2', \mathcal{R}_3$:

$$\begin{bmatrix} \frac{-t+1+t^2}{t^2} & \frac{(-1+t)^2}{t^3} & -\frac{1-t+t^3}{t^3} \\ 0 & -\frac{-t+1+t^2}{t^2} & \frac{-t+1+t^2}{t^2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that rank$(\phi(w) - id) = 2$ in $\Lambda/(\Delta_{8_{20}}(t))$ so that 8_{20} admits a non-trivial coloring by $Q = \Lambda/(\Delta_L(t))$, e.g., $(1 - t + t^2, 0, 0)$. Note that, in $\Lambda/(1 - t + t^2)$, rank$(\phi(w) - id) = 1$ and that $(1, 0, 0)$ is a non-trivial coloring of 8_{20} by $Q = \Lambda/(1 - t + t^2)$, and hence every coloring of 8_{20} by $Q = \Lambda/(1 - t + t^2)$ is of the form $\alpha(t)(1, 0, 0) + \beta(t)(1, 1, 1) = (\alpha(t) + \beta(t), \beta(t), \beta(t))$.

Non-Trivial Coloring Table.
The table in the last page is a list of non-trivial colorings, in which we used the braid notations and Alexander polynomials in *KnotInfo: Table of Knot Invariants*.

Non-Trivial Coloring Table.

Name	non-trivial coloring by Alexander quandle $\Lambda/(\Delta_L(t))$
3_1	$(1, 0)$
4_1	$(-1+t, -1+2t, 0)$
5_1	$(1, 0)$
5_2	$(3t-2, 4-2t, 0)$
6_1	$(2t-1, -1+t, 3t-2, 0)$
6_2	$(t^3-2t^2+2t-1, 2t^3-2t^2+2t-1, 0)$
6_3	$(-t^2+2t-1, -2t^2+2t-1, 0)$
7_1	$(1, 0)$
7_2	$(11t-12, -5t+3, 6t-9, 0)$
7_3	$(t^3-t^2+3t-2, -2t^3+2t^2-2t+4, 0)$
7_4	$(-53t+76, 56t-32, -96t+128, 0)$
7_5	$(-t^3+2t^2-2t, 4t-4t^2-4+2t^3, 0)$
7_6	$(3t^3-6t^2+5t-1, 4t^3-6t^2+5t-1, -4t^2-1+4t+2t^3, 0)$
7_7	$(2t^3-7t^2+5t-1, 2t^3-8t^2+5t-1, -6t^2+2t^3-1+4t, 0)$
8_1	$(2t-2, 3t-2, -1+t, 4t-3, 0)$
8_2	$(t^5-2t^4+2t^3-2t^2+2t-1, t^5-2t^4+2t^3-2t^2+2t-1, 0)$
8_3	$(13t-12, 21t-12, 5t-4, 29t-20, 0)$
8_4	$(5t^3-t^2+3t-2, t^3-t^2+3t-2, 9t^3-9t^2+11t-6, 0)$
8_5	$(t^3-t^2+t-1, 2t^3-t^2+t-1, 0)$
8_6	$(2t^3-2t^2+2t-1, t^3-2t^2+2t-1, 3t^3-4t^2+4t-2, 0)$
8_7	$(-t^4+2t^3-2t^2+2t-1, -2t^4+2t^3-2t^2+2t-1, 0)$
8_8	$(-2t^2+2t-1, -t^2+2t-1, -3t^2+4t-2, 0)$
8_9	$(t^3-2t^2+2t-1, 2t^3-2t^2+2t-1, 0)$
8_{10}	$(-t^4+2t^3-3t^2+2t-1, -2t^4+3t^3-3t^2+2t-1, 0)$

8_{11}	$(3t^3-5t^2+5t-2, 2t^3-5t^2+5t-2, 4t^3-6t^2+5t-2, 0)$
8_{12}	$(t^3-4t^2+3t, t^3-4t^2+2t, -4t^2+4t+t^3-1, 2t^3-8t^2+6t-1, 0)$
8_{13}	$(-3t^3+3t^2-5t+2, -3t^3+7t^2-5t+2, -3t^3-t^2+3t-2, 0)$
8_{14}	$(3t^3-5t^2+5t-2, 2t^3-5t^2+5t-2, 4t^3-7t^2+6t-2, 0)$
8_{15}	$(5t-3t^2-3, (-1+t)(3t^2-5t+3), 0, 0), (t(5t-3t^2-3), 5t-3t^2-3, 0, 0)$
8_{16}	$(-2t^4+3t^3-4t^2+3t-1, t^5-4t^4+5t^3-5t^2+3t-1, 0)$
8_{17}	$(2t^3-3t^2+3t-1, -t^4+4t^3-4t^2+3t-1, 0)$
8_{18}	$(2t-1, t^3-2t^2+3t-1, 0)$
8_{19}	$(t, -t^2+t+1, 0)$
8_{20}	$(1-t+t^2, 0, 0)$