New Filtrations on Homology Cylinder Groups

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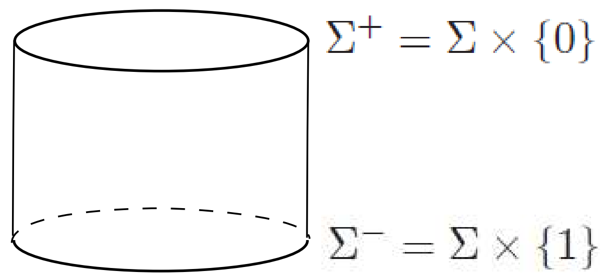
Definition of homology cylinders

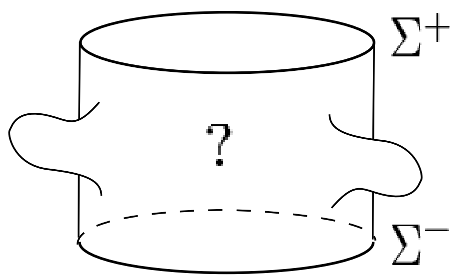
Σ: an oriented, compact surface

Cylinder:



$\Sigma \times I$

Homology cylinder:



M

$$H_*(\Sigma^{\pm}) \simeq H_* M$$

Definition of homology cylinders

A homology cylinder (M, i_+, i_-) over a surface Σ is a 3-manifold M together with injections $i_+,\ i_- : \Sigma \to \partial M$ satisfying the following:

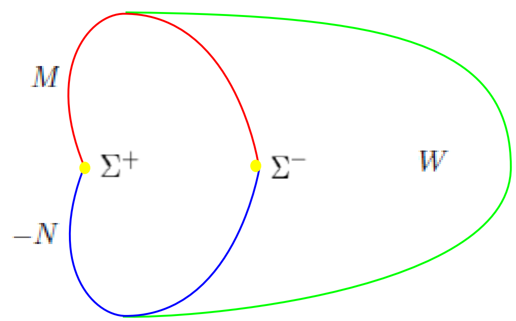
1. i_+ is orientation preserving and i_- is orientation reversing.
2. $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$.
3. $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$
4. $i_+, i_- : H_*(\Sigma) \to H_*(M)$ are isomorphisms.

e.g. Knot exteriors (surface=annulus)
string links (surface=disk with punctures)
elements of mapping class group...

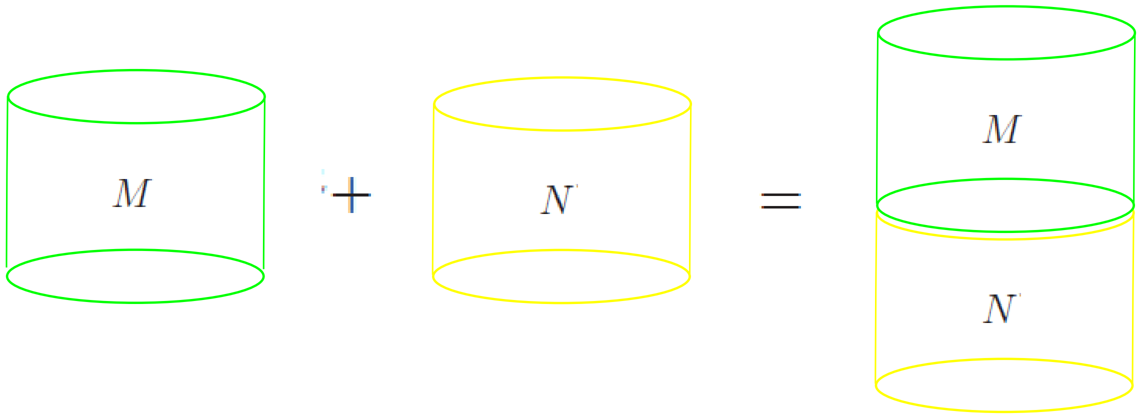
Definition of homology cobordism

M, N:homology cylinders over $\Sigma_{g,n}$

M and N are called homology cobordant if there is a 4-manifold W such that $M \cup_{\partial} (-N) = \partial W$ and the inclusion induced maps $H_*(M) \to H_*(W)$ and $H_*(N) \to H_*(W)$ are isomorphisms, or equivalently, $H_*(W, M) = H_*(W, N) = 0$.



Homology cobordism is an equivalence relation on $\mathcal{C}_{g,n}$: the set of homology cylinders over surface $\Sigma_{g,n}$ with genus g and n punctures. Let $\mathcal{H}_{g,n}$ be the set of equivalence classes in $\mathcal{C}_{g,n}$ under homology cobordism.



Stack

Then $\mathcal{H}_{g,n}$ has a group structure under stacking as binary operation.

$\mathcal{H}_{g,n}$: homology cylinder group over $\Sigma_{g,n}$

Definition of homology cobordism

M and N are called homology cobordant if there is a 4-manifold W such that $M \cup_\partial (-N) = \partial W$ and $H_1(W, M) = H_1(W, N) = 0$ and $H_2(W, M) = H_2(W, N) = 0$.

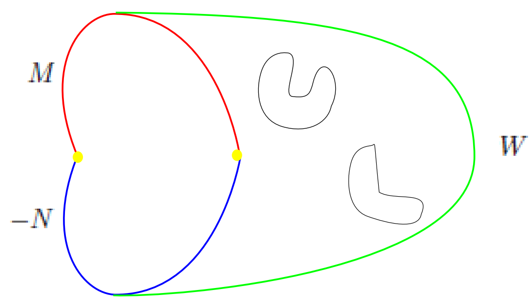
Definition of homology cobordism

M and N are called homology cobordant if there is a 4-manifold W such that $M \cup_\partial (-N) = \partial W$ and such that $H_1(W, M) = H_1(W, N) = 0$ and $H_2(W, M) = H_2(W, N) = 0$.

Idea of our construction of new filtration.
By relaxing the H_2 condition, we could define coarser equivalence relations on $\mathcal{H}_{g,n}$.

Idea of new equivalence relation

Cappell and Sheneson:
if there is a 4-manifold W bounding $M \cup_\partial (-N)$ such that $H_1(W, M) = H_1(W, N) = 0$ and there are embedded 2-spheres in W which generate a Lagrangian subgroup of $H_2(W, M) = H_2(W, N)$, then by surgery along these spheres, a 4-manifold W' with $H_2(W', M) = H_2(W', N) = 0$ is obtained, so that M and N are homology cobordant.



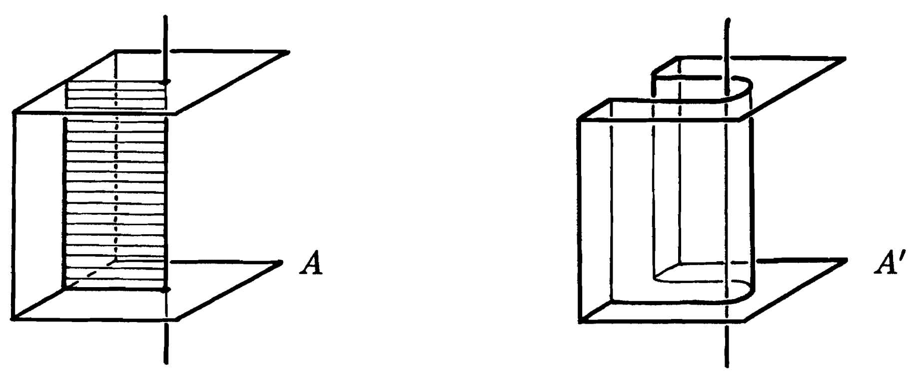
Whitney tower: approximation of embedding of surfaces.

Whitney Tower

W: an orientable 4-manifold

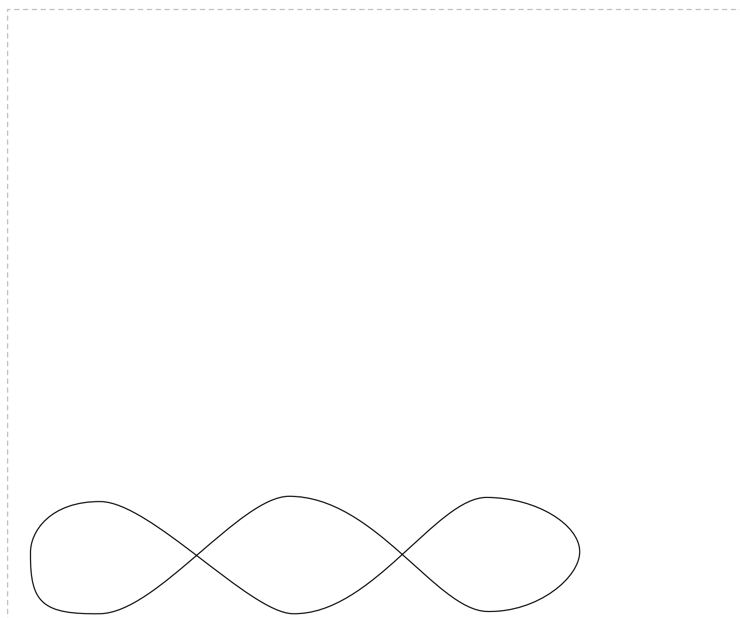
$\iota: S \looparrowright W$, an immersion of a surface S.

For two self-intersection points of ι, if there is a framed embedded disk looks like below, called a embedded Whitney disk, then we can eliminate these self-intersection points by Whitney move.

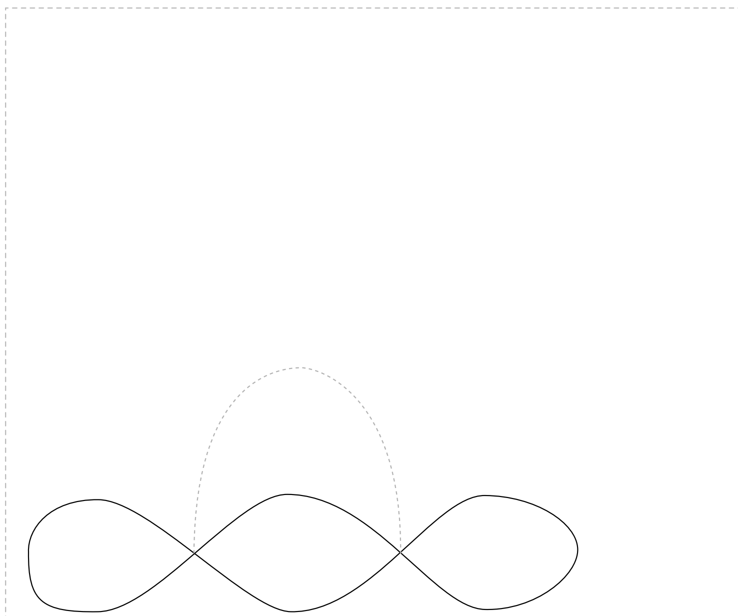


Reference: [Milnor, Lectures on h-cobordism theorem]

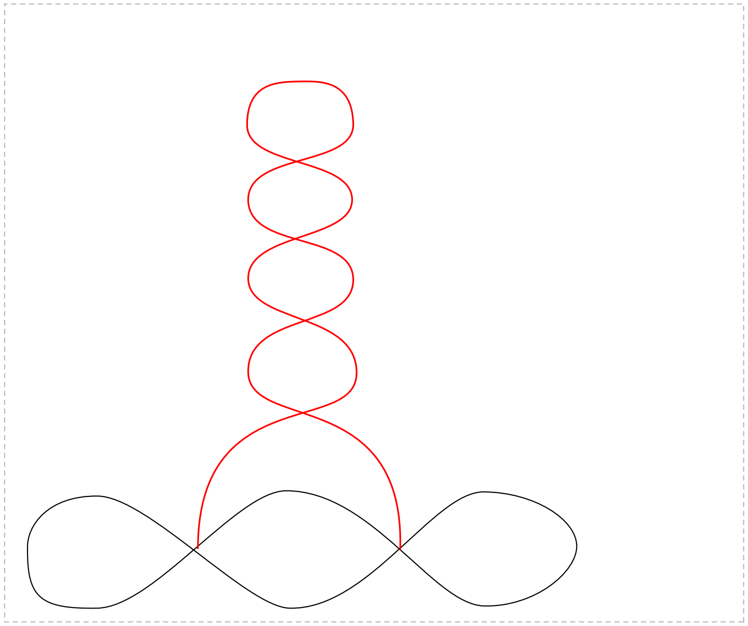
Here is an immersed sphere $S \looparrowright W$ in a 4-manifold W.



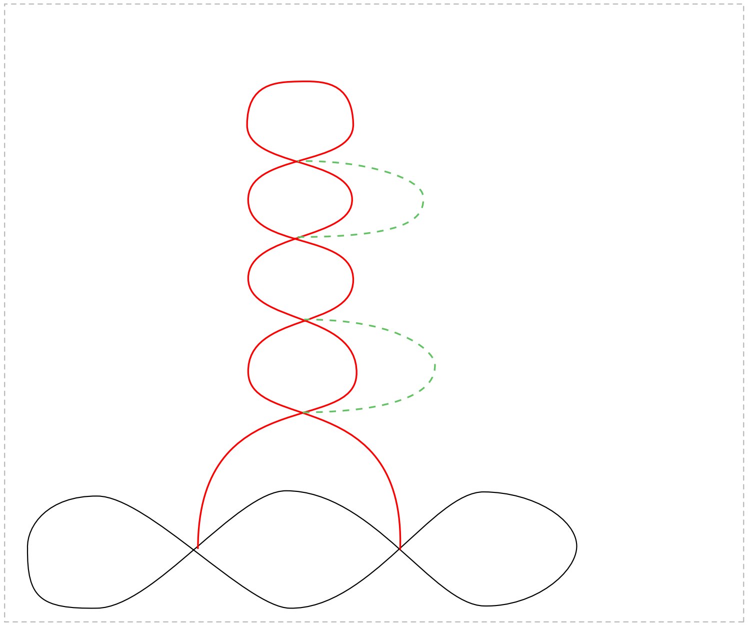
If all intersection points are paired up by embedded Whitney disks, then by Whitney move, $S \looparrowright W$ can be isotoped to an embedding.



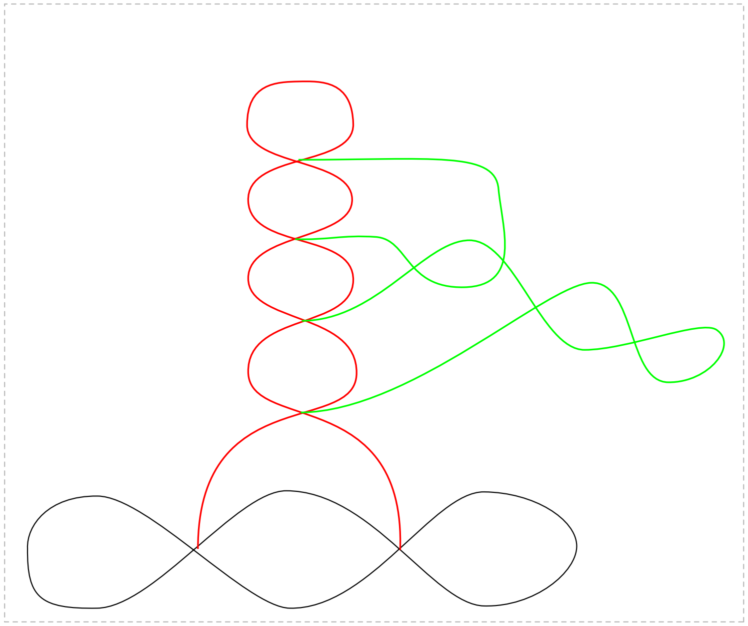
If all self-intersection points of S can be paired up such that each pair has an immersed Whitney disk, then S is said to have a Whitney tower of height 1.



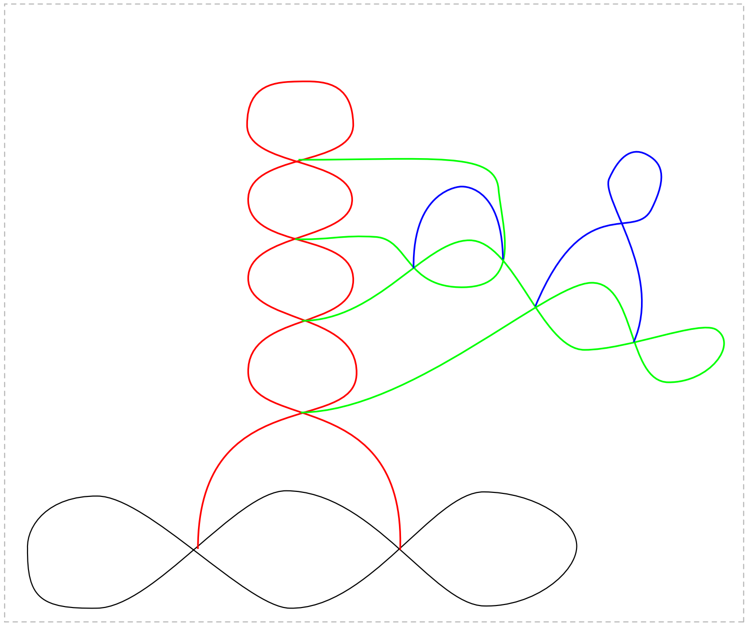
If all intersection points on height 1 Whitney disks are paired up by embedded Whitney disks, then by Whitney move, $S \looparrowright W$ can be isotoped to an embedding.



If all intersection points on height 1 Whitney disk can be paired up such that each pair has an immersed Whitney disk, then S is said to have a Whitney tower of height 2.

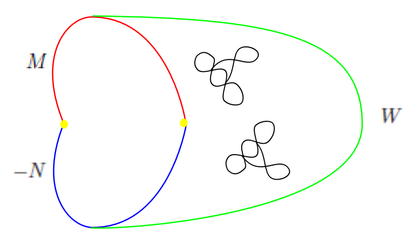


Inductively, we can define a Whitney tower of height h for any $h \in \mathbb{N}$.



Height h Whitney tower cobordism [Cha]

Two homology cylinders M and N over Σ are called height h Whitney tower cobordant($h = 0, 1, 2, \cdots$) if there exists a 4-manifold W such that $\partial W = M \cup_\partial (-N)$ such that $H_1(W, M) = H_1(W, N) = 0$, and there are immersed 2-spheres bounding height h Whitney tower in W which generate a Lagrangian subgroup of $H_2(W, M)$.



Let $\mathcal{W}_h \subset \mathcal{H}_{g,n}$ be the subset of homology cylinders which are height h Whitney tower cobordant to the trivial homology cylinder.

Theorem. *For $n > 0$, $\mathcal{W}_h$ is well-defined and is a subgroup of $\mathcal{H}_{g,n}$.*

h-solvable cobordism [Cha]

Two homology cylinders M and N over Σ are called h-solvable cobordant if there exists a 4-manifold W bounding $M \cup (-N)$ and

1. $H_1(M) \simeq H_1(W)$, $H_1(N) \simeq H_1(W)$, and
2. there are $l_1, \cdots, l_r$ and $d_1, \cdots, d_r \in H_2(W, \mathbb{Z}[\pi/\pi^{(h)}])$, where $\pi = \pi_1(W)$, such that $2r$ =rank$(H_2(W, M))$, $\lambda(l_i, l_j) = 0$ and $\lambda(l_i, d_j) = \delta_i^j$ for $i, j = 1, \cdots, r$, and the image of $l_1, \cdots, l_r$ and $d_1, \cdots, d_r$ into $H_2(W, M)$ forms a basis.

Let $\mathcal{S}_h \subset \mathcal{H}_{g,n}$ be the subset of homology cylinder which is h-solvable cobordant to the trivial homology cylinder. $(\mathcal{W}_h \subset \mathcal{S}_h)$

Theorem. *For $n > 0$, $\mathcal{S}_h$ is well-defined and is a subgroup of $\mathcal{H}_{g,n}$.*

So we have two filtrations of $\mathcal{H}_{g,n}$:

$$0 \subset \cdots \subset \mathcal{W}_h \subset \cdots \subset \mathcal{W}_1 \subset \mathcal{W}_0 \subset \mathcal{H}$$

$$0 \subset \cdots \subset \mathcal{S}_h \subset \cdots \subset \mathcal{S}_1 \subset \mathcal{S}_0 \subset \mathcal{H}$$

Question. Are they nontrivial filtration? What is the structure of this filtration?

Our Results

Theorem. *For $n > 0$ and $(g, n) \neq (0, 1), (0, 2)$, two filtrations are nontrivial, i.e. for all h, $\mathcal{W}_h/\mathcal{W}_{h+1}$ and $\mathcal{S}_h/\mathcal{S}_{h+1}$ are not trivial. Moreover, $\mathcal{S}_h/\mathcal{S}_{h+1}$ contains an infinitely generated subgroup.*

Remark.

1. For $(0, 1)$ case, $\mathcal{H}_{g,n}$ is the same with the group of cobordism classes of integral homology 3-spheres, which is a trivial group in topological category.

2. For $(0, 2)$ case, $\mathcal{H}_{g,n}$ corresponds to knot concordance group in integral homology 3-spheres. The same conclusion is obtained by [Cochran, Orr, Teichner, 2003], [Cochran, Teichner, 2007], [Cochran, Harvey, Leidy, 2009], and [Horn,2012].

Sketchy of proof of the first part

First, generate homology cylinders in $\mathcal{W}_h$ or $\mathcal{S}_h$ by satellite construction on the trivial homology cylinder along well-chosen curve, and using Cochran-Teichner's knot.

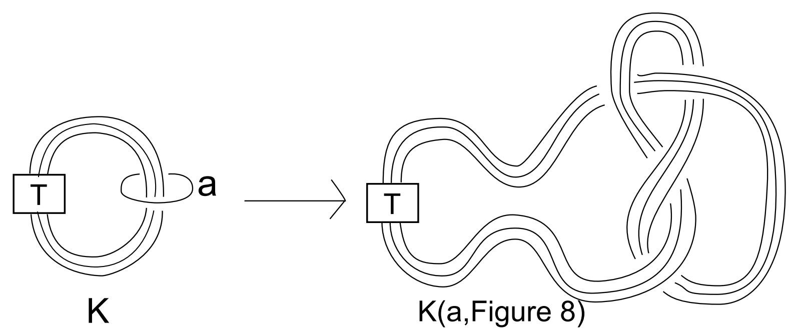


Figure 1: Satellite construction (on knot)

Theorem (Amenable signature theorem, Cha)**.** *Let* W *be an* $h+1$*-solvable cobordism between* M *and* N *and* G *be an amenable group lying in Strebel's class* $D(R)$*,* $R = \mathbb{Z}/p$ *or* $\mathbb{Q}$*, and* $G^{(h+1)} = 0$*. Suppose* $\phi\colon \pi_1(M \cup (-N)) \to G$ *factors through* $\pi_1 W$*, then the Cheeger-Gromov invariant* $\rho(M \cup (-N), \phi)$ *vanishes.*

Using this theorem and Cheeger-Gromov estimate, we get the desired conclusion.

For the second assertion about infinitely generated free abelian subgroup, we need more argument, namely a new Dwyer type theorem for mixed-coefficient commutator series.