Twisted biquandles and invariants of twisted links

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Abstract: A biquandle is a set with two binary operations satisfying certain conditions coming from Reidemeister moves. We consider an additional structure related to twisted knots. The number of colorings is an invariant of a twisted knot. There is also a notion of the fundamental twisted biquandle of a twisted knot.

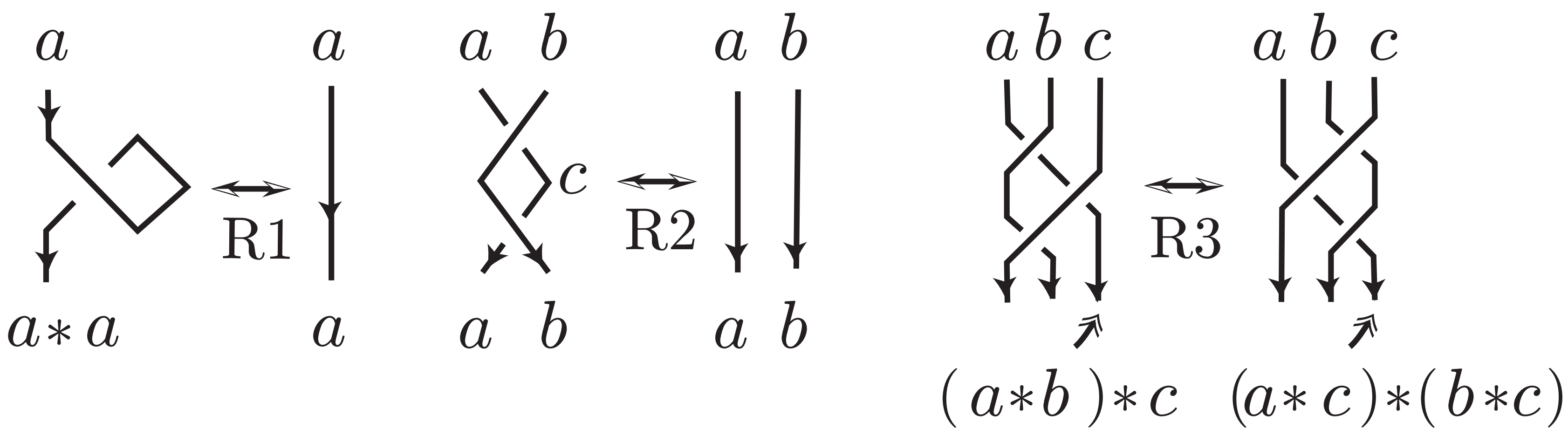
Definition (Joyce '82, Matveev '82, Fenn-Rourke '92)

A *quandle* is a set X with a binary operation $*$ such that

(Q1) $\forall a \in X$, $a * a = a$,

(Q2) $\forall a, b \in X$, $\exists! c \in Q$ with $c * b = a$, and

(Q3) $\forall a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.



Definition (Fenn, Jordan-Santana, Kauffman '04)

A *biquandle* is a set X with two binary operations $*$ and $\circ$ satisfying a certain condition.

notation: $x * y =: x^y$ and $x \circ y =: x_y$
Put $R : X^2 \to X^2$, $R(a, b) = (b_a, a^b)$.

Definition (Fenn, Jordan-Santana, Kauffman '04)

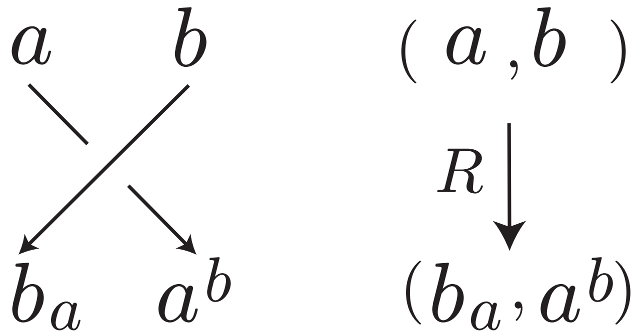
A *biquandle* is a pair (X, R) of a set X and a bijection $R : X^2 \to X^2$ satisfying the following.

(B1) $(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R)$.

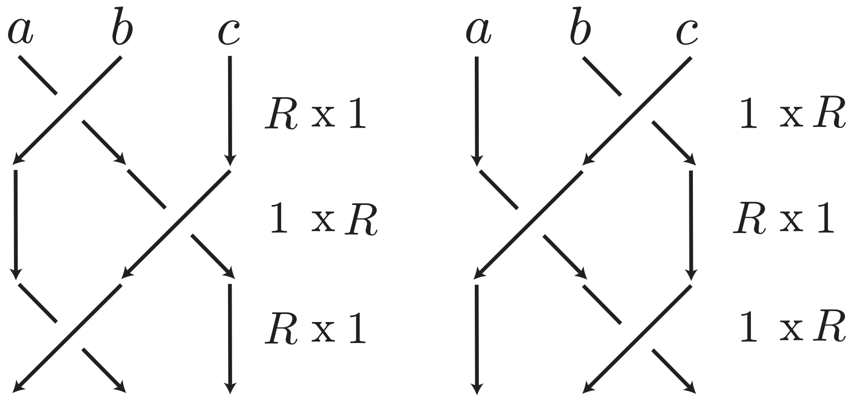
(B2) For $a, b \in X$, let $f_a : X \to X$ and $f^b : X \to X$ be maps defined by $f_a(x) = p_1 R(a, x)$ and $f^b(x) = p_2 R(x, b)$. Then, $\forall a, b \in X$, f_a and f^b are bijections.

(B3) $\forall a, b \in X$,
$(f_a)^{-1}(a) = f^{(f_a)^{-1}(a)}(a)$ and $(f^b)^{-1}(b) = f_{(f^b)^{-1}(b)}(b)$.

$$\boldsymbol{R : X^2 \to X^2,\ R(a,b) = (b_a, a^b)}$$



(B1) $\boldsymbol{(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R)}$



(B1) ⇔ R3-move, (B2) ⇔ R2-move, (B3) ⇔ R1-move

Definition

A *biquandle* is a pair (X, R) of a set X and a bijection $R : X^2 \to X^2$ satisfying the following.

(B1) $(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R)$.

(B2′) $\exists!$ a unique bijection $S : X^2 \to X^2$ s.t. $\forall x_1, \dots x_4 \in X$,

$$S(x_1, x_3) = (x_2, x_4) \Longleftrightarrow R(x_1, x_2) = (x_3, x_4).$$

(B3′) $\exists$ a bijection $s : X \to X$ s.t. $\forall x \in X$,

$$R(x, s(x)) = (x, s(x)).$$

We call $S : X^2 \to X^2$ the *sideways operation* and $s : X \to X$ the *shift operation* of R.
($s : X \to X$ is unique.)

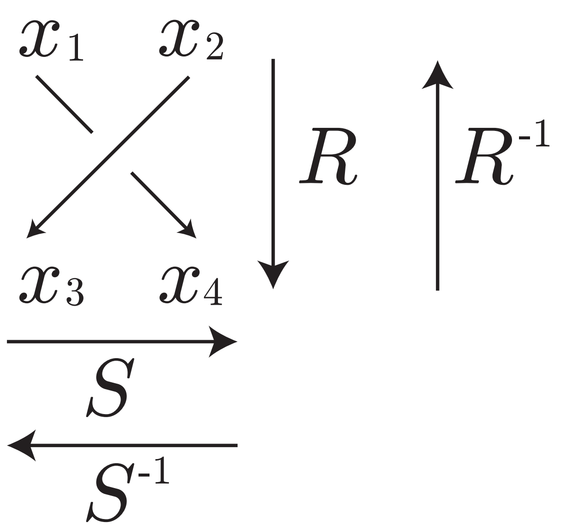
Theorem

The two definitions are equivalent. Precisely speaking, (B2) ⇔ (B2′). If (B2) is satisfied, then (B3) ⇔ (B3′).

(B2) For $a, b \in X$, let $f_a : X \to X$ and $f^b : X \to X$ be maps defined by $f_a(x) = p_1 R(a, x)$ and $f^b(x) = p_2 R(x, b)$. Then, $\forall a, b \in X$, f_a and f^b are bijections.

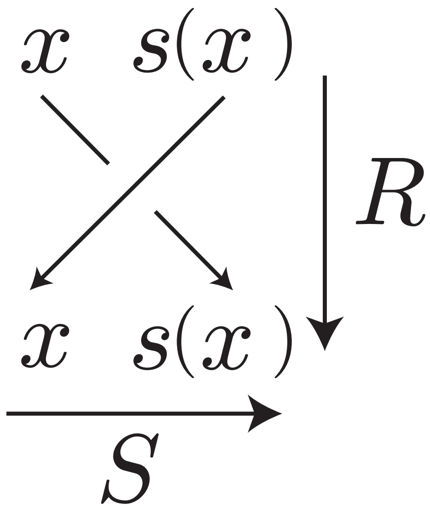
(B2′) $\exists!$ a unique bijection $S : X^2 \to X^2$ s.t. $\forall x_1, \dots x_4 \in X$,

$$S(x_1, x_3) = (x_2, x_4) \iff R(x_1, x_2) = (x_3, x_4).$$



(B3) $\forall a, b \in X$,
$$(f_a)^{-1}(a) = f^{(f_a)^{-1}(a)}(a) \quad \text{and} \quad (f^b)^{-1}(b) = f_{(f^b)^{-1}(b)}(b).$$

(B3′) $\exists$ a bijection $s : X \to X$ s.t. $\forall x \in X$,
$$R(x, s(x)) = (x, s(x)).$$



Proof. (B3) $\Rightarrow$ (B3′).
$$s(x) = (f_x)^{-1}(x) \quad \text{and} \quad s^{-1}(y) = (f^y)^{-1}(y).$$

Examples ([Bartholomew and Fenn, JKTR 20(2011)])

(1) Alexander biquandle (Sawollek) $\boldsymbol{X}$: a $\mathbb{Z}[\boldsymbol{\lambda}^{\pm 1}, \boldsymbol{\mu}^{\pm 1}]$-module.

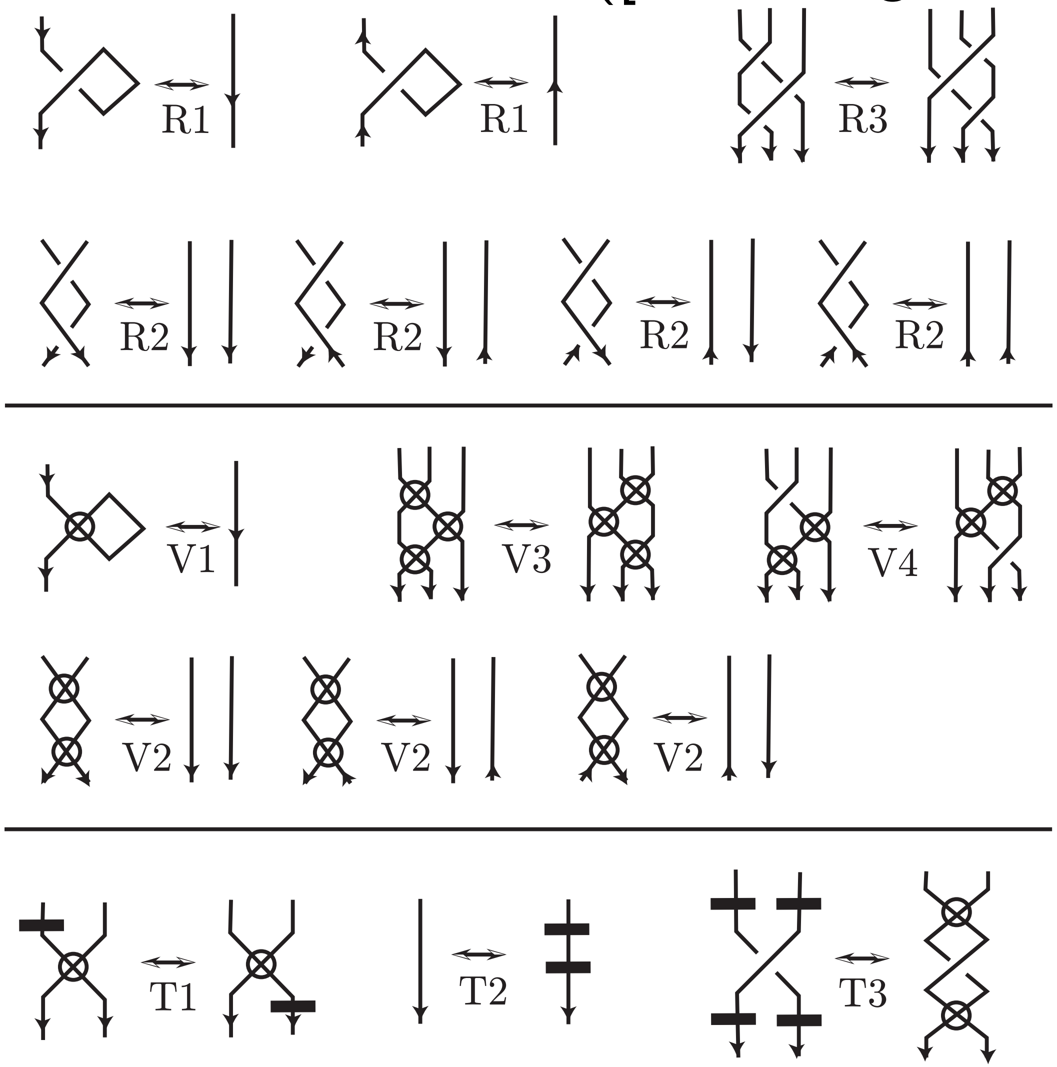
$$\boldsymbol{a^b = \lambda a + (1 - \lambda\mu)b, \quad a_b = \mu a.}$$

(2) Budapest biquandle (Fenn) $\boldsymbol{H}$: the quaternion algebra with standard generators $\boldsymbol{1, i, j, k}$. $\boldsymbol{X}$: a left $\boldsymbol{H}$-module.

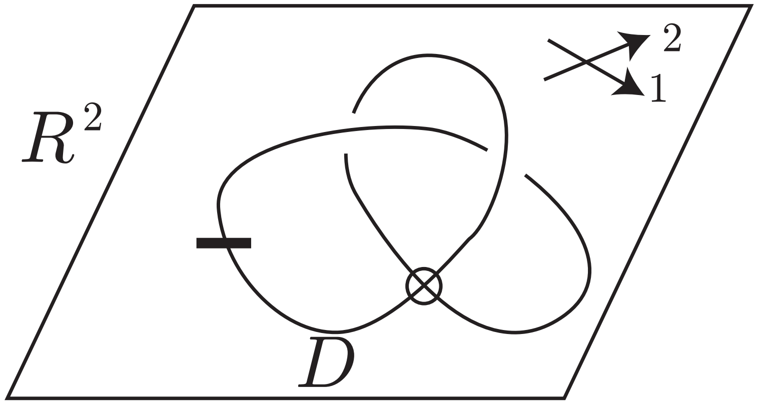
$$\boldsymbol{a^b = ja + (1 + i)b, \quad a_b = -ja + (1 + i)b.}$$

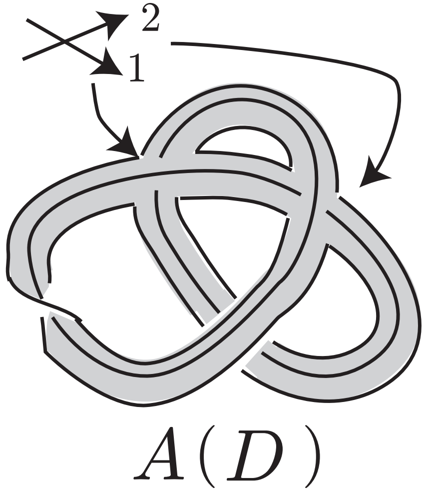
(3) (Wada) $\boldsymbol{X}$: a group, $\boldsymbol{R(a, b) = (a^2b, b^{-1}a^{-1}b)}$.
(4) (Silver and Williams) $\boldsymbol{X}$: a group with a $\mathbb{Z}^2$ action, $\boldsymbol{a \longrightarrow a \cdot x}$, $\boldsymbol{a \longrightarrow a \cdot y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ is a basis for $\mathbb{Z}^2$.
$\boldsymbol{R(a, b) = (b \cdot y, (b \cdot xy)^{-1}(a \cdot x)b)}$.

A *twisted link diagram* is a virtual link diagram which may have some bars on edges. Two diagrams are *equivalent* if there is a sequence of local moves depicted in Figure. The equivalence class of a twisted link diagram is called a *twisted link*. ([M. Bourgoin, AGT 8(2008)])



For a twisted link diagram D,
we have a generalized abstract link diagram $A(D)$. Attaching disks to the boundary, we have a link diagram on a closed surface.





Theorem (N. and S. K. (2000, JKTR), Carter-Kamada-Saito (2002), Bourgoin (2008))

$$\{\textit{virtual links}\} \Leftrightarrow \{\textit{abstract links}\} \Leftrightarrow \{\textit{links in } F \times I\}/\sim$$

$$\{\textit{twisted links}\} \Leftrightarrow \{\textit{gen. abstract links}\} \Leftrightarrow \{\textit{links in } F \tilde{\times} I\}/\sim$$

v- and t-Strucures on biquandles

Let (X, R) be a biquandle.

Definition

A bijection $V : X^2 \to X^2$ is a *v-structure* of (X, R) if the following conditions are satisfied.

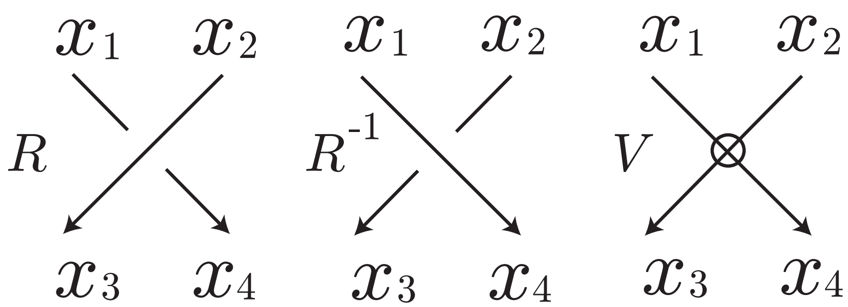
(1) (X, V) is a biquandle.

(2) $V^2 = 1 : X^2 \to X^2$.

(3) $(V \times 1)(1 \times V)(R \times 1) = (1 \times R)(V \times 1)(1 \times V)$.

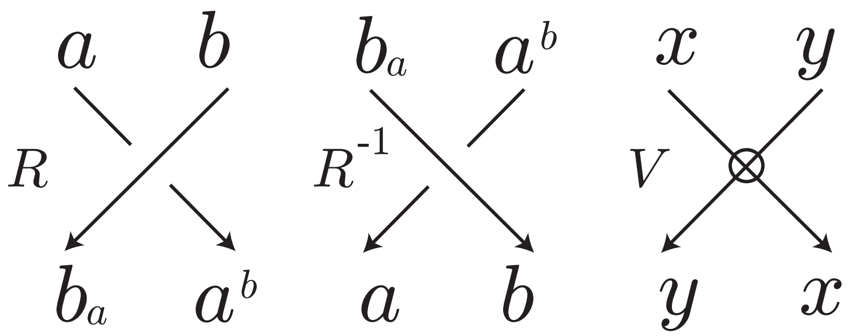
We call (X, R, V) a *v-structured biquandle*.

A v-structure is used for colorings of virtual link diagrams.



Example

Let (X, R) be a biquandle. Let V be the transposition $\tau : X^2 \to X^2, (x, y) \mapsto (y, x)$. It is a v-structure. Biquandles with this v-structure are used for colorings of virtual link diagrams in many articles.



Example

(Kauffman and Manturov '05) Let (X, R) be a biquandle. Let $f : (X, R) \to (X, R)$ be an automorphism. Let $V : X^2 \to X^2$ be a map defined by $V(x_1, x_2) = (f^{-1}x_2, fx_1)$. It is a v-structure. A biquandle with this structure is called a *virtual biquandle*.

We introduce the notion of a t-structure, or a vt-structure, which is related to twisted links.

Definition

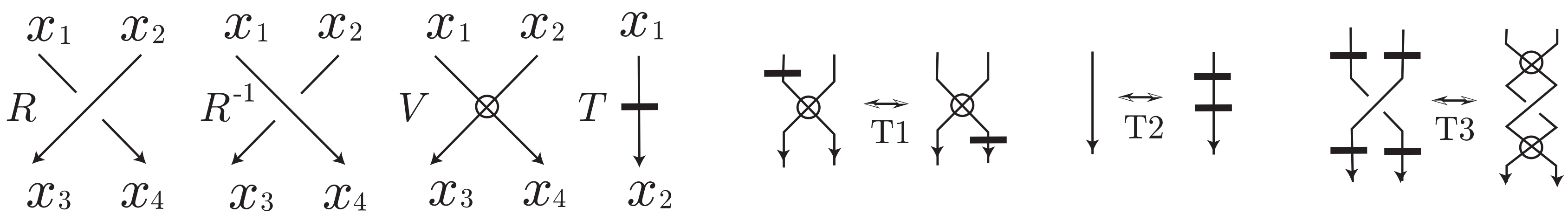
Let (X, R, V) be a v-structured biquandle. A bijection $T : X \to X$ is a *t-structure* of (X, R, V) if the following conditions are satisfied.

(1) $T^2 = 1$.

(2) $V(T \times 1) = (1 \times T)V$.

(3) $(T \times T)R(T \times T) = VRV$.

We call (X, R, V, T) a *vt-structured biquandle*. We also call (V, T) a *vt-structure* of (X, R).



Colorings

D: a twisted link diagram. The *edges* of D mean the connected arcs obtained when all the real crossings, virtual crossings and bars are removed. $E(D) :=$ the set of edges of D.

Definition

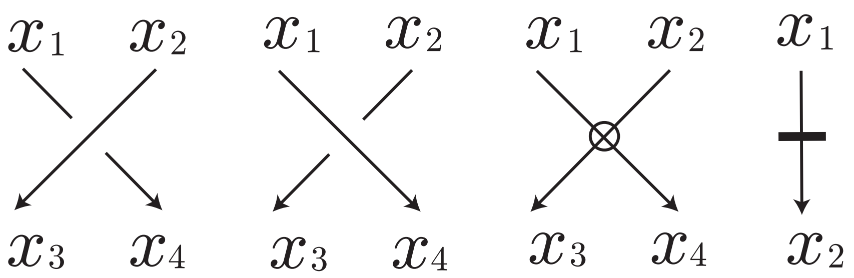
A *coloring* of D by (X, R, V, T) is a map $E(D) \to X$ such that if x_1, x_2, x_3, x_4 are as in the Figure then

(1) $R(x_1, x_2) = (x_3, x_4)$ when v is a positive crossing,

(2) $R^{-1}(x_1, x_2) = (x_3, x_4)$ when v is a negative crossing,

(3) $V(x_1, x_2) = (x_3, x_4)$ when v is a virtual crossing, and

(4) $T(x_1) = x_2$ when v is a bar.



Theorem (Bartholomew and Fenn '10)

If $\boldsymbol{D}$ and $\boldsymbol{D'}$ are virtual link diagrams representing the same virtual link, then there is a bijection between the set of colorings of $\boldsymbol{D}$ by a v-structured biquandle $(\boldsymbol{X}, \boldsymbol{R}, \boldsymbol{V})$ and that of $\boldsymbol{D'}$.

This is generalized to twisted link diagrams.

Theorem

If $\boldsymbol{D}$ and $\boldsymbol{D'}$ are twisted link diagrams representing the same twisted link, then there is a bijection between the set of colorings of $\boldsymbol{D}$ by a vt-structured biquandle $(\boldsymbol{X}, \boldsymbol{R}, \boldsymbol{V}, \boldsymbol{T})$ and that of $\boldsymbol{D'}$.

Therefore the numbers of colorings by $(\boldsymbol{X}, \boldsymbol{R}, \boldsymbol{V}, \boldsymbol{T})$ is an invariant of a twisted link.
We want examples of vt-structured biquandles $(\boldsymbol{X}, \boldsymbol{R}, \boldsymbol{V}, \boldsymbol{T})$.

§3. Construction

Let (X_0, R_0) be a biquandle, $R_0(a,b) = (b_a, a^b)$. Let f and g be automorphisms of (X_0, R_0) with $f^2 = 1$ and $fg = gf$.

Theorem

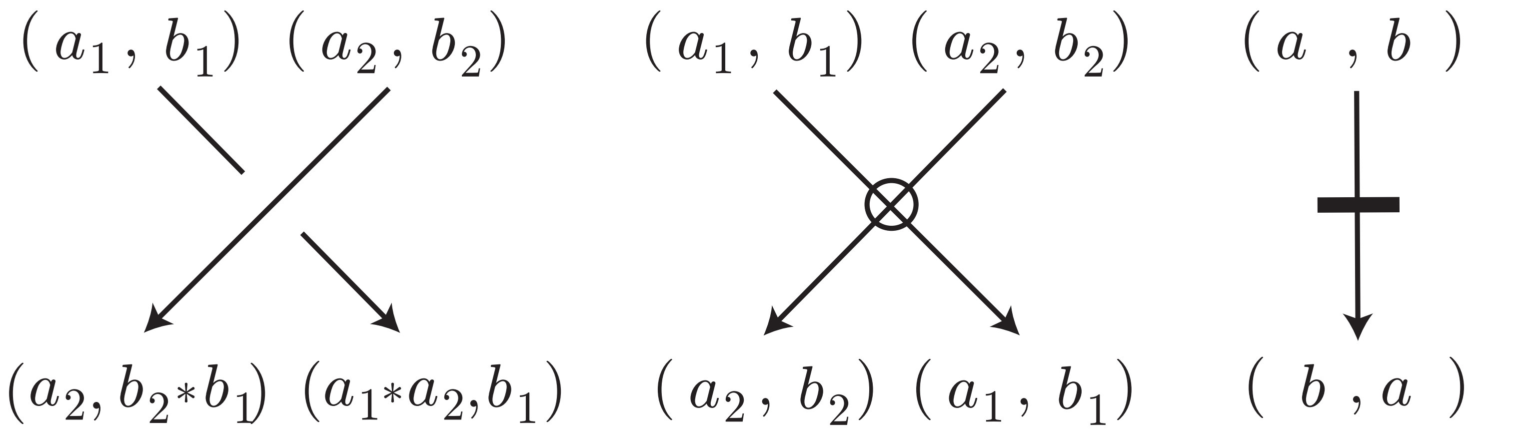
Let $X = X_0 \times X_0$. We have a vt-structured biquandle (X, R, V_f, T_g) with

$$\begin{aligned} R((a_1,b_1),(a_2,b_2)) &= ((a_{2a_1}, {b_2}^{b_1}), ({a_1}^{a_2}, b_{1b_2})) \\ V_f((a_1,b_1),(a_2,b_2)) &= ((fa_2, fb_2),(fa_1, fb_1)), \\ T_g(a,b) &= (g^{-1}b, ga). \end{aligned}$$

We call (X, R, V_f, T_g) a *twisted product* of (X_0, R_0). When $f = g = 1$, we call it the *standard twisted product* of (X_0, R_0).

When (X_0, R_0) is derived from a quandle $Q = (Q, *)$, i.e., $R_0(a, b) = (b, a * b)$, we denote by $\mathcal{B}(Q)$, the standard twisted product of (X_0, R_0). Namely, $X = Q \times Q$ and

$$\begin{aligned} R((a_1, b_1), (a_2, b_2)) &= ((a_2, b_2 * b_1), (a_1 * a_2, b_1)), \\ V((a_1, b_1), (a_2, b_2)) &= ((a_2, b_2), (a_1, b_1)), \text{ and} \\ T(a, b) &= (b, a). \end{aligned}$$



§4. Invariants

For a vt-structured biquandle (X, R, V, T), the number of colorings of a diagram D is a twisted link invariant.
In particular, for a quandle Q, the number of colorings by the standard twisted product $\mathcal{B}(Q)$ is a twisted link invariant.

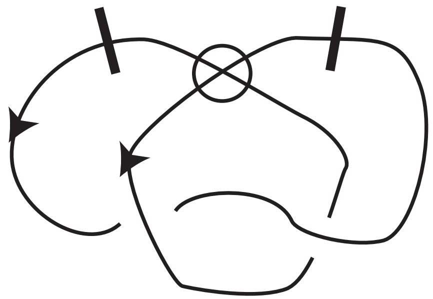
Theorem

Let D be a twisted link diagram. If $D \cong D'$ for a virtual link diagram D', then the number of $\mathcal{B}(Q)$-colorings of D is the product of the number of upper Q-colorings of D' and that of lower Q-colorings of D'.

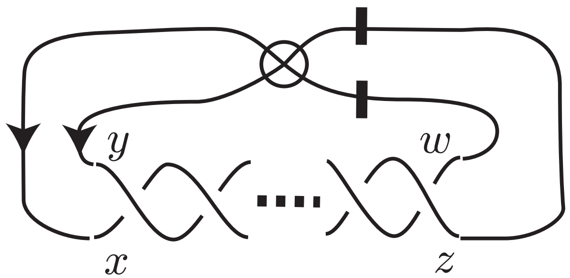
Corollary

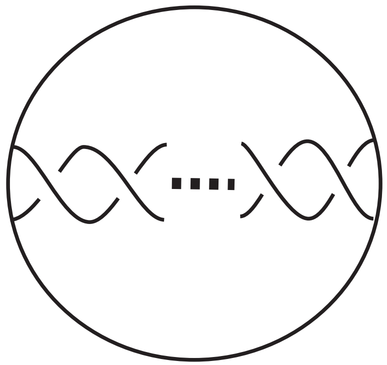
Let Q be a finite quandle with n elements. If the number of $\mathcal{B}(Q)$-colorings of a twisted link diagram D is less than n^2, then D does not represent a virtual link.

Using the coloring invariant, we see that *Bourgoin's twofoil* is not a virtual knot.



Let F_m be a *non-orientable virtual m-foil*, where m is the number of the real crossings $(m \geq 1)$. When $m = 2$, it is Bourgoin's twofoil.





Theorem

(1) *For $m > m' \geq 1$, F_m and $F_{m'}$ represent distinct twisted links.*

(2) *For $m \geq 1$, F_m does not represent a virtual link.*