Free degrees of homeomorphisms on compact surfaces

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$f\colon M \to M$: a self homeomorphism on a compact surface M;

$\mathfrak{fr}(f) = \max\{m \mid \cup_{j=1}^{m-1} \mathrm{Fix}(f^j) = \emptyset\}$: the free degree of f;

$\mathfrak{fr}(M) = \max\{\mathfrak{fr}(f) \mid f \in Home(M)\}$: the free degree of all homeomorphisms on M;

$\mathfrak{fr}^+(M) = \max\{\mathfrak{fr}(f) \mid f \in Home^+(M)\}$: the free degree of all orientation preserving homeomorphisms on M;

$o(M)$: the maximum of the orders of the periodic homeomorphisms on M;

F_g: oriented closed surface of genus g;

N_g: non-orientable closed surface of genus g;

$F_{g,b}$: F_g with b holes;

$N_{g,b}$: N_g with b holes;

$L(f)$: the Lefschetz number of f;

$N(f)$: the Nielsen number of f;

Background

J. Nielsen [5] studied $\mathfrak{fr}^+(F_g)$ in nineteen forties, showing that

$$\mathfrak{fr}^+(F_g) = \begin{cases} 2 \; or \; 3, & \text{if } g = 2, \\ 2g-2, & \text{if } g > 2. \end{cases}$$

W. Dicks and J. Llibre [1] determined $\mathfrak{fr}^+(F_2) = 2$ in 1996.

In nineteen nineties, S. Wang [7] obtained:

$$\mathfrak{fr}(F_g) = \begin{cases} 4, & \text{if } g = 2, \\ 2g-2, & \text{if } g > 2. \end{cases}$$

Our main result is:

Theorem

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

Main tool

Proposition

$$\mathfrak{fr}(f) \leq \min\{n \mid N(f^n) > 0\} \leq \min\{n \mid L(f^n) \neq 0\}.$$

Thurston's classification theorem of surface homeomorphisms.

"Standard" homeomorphisms introduced in B. Jiang and J. Guo [4].

Periodic homeomorphisms

Lemma (1) $o(F_{g,b}) \le o(F_g)$ for all b; (2) $o(N_{g,b}) \le o(N_g)$ for all b.

Lemma Let f be a self homeomorphism on a connected compact surface M which is homotopic to a periodic map. If $\chi(M) \neq 0$, then there is a positive integer $n \le o(M)$ such that $N(f^n) = 1$, and hence the free degree $\mathfrak{fr}(f)$ of f is no more than $o(M)$.

Theorem [S. Wang 1991] (1) $o(F_g) = 4g + 3 + (-1)^g$ for all $g \ge 2$.

(2) $o(N_g) = 2g - 1 + (-1)^g$ for all $g \ge 3$.

Theorem (1) Let $f\colon F_{g,b} \to F_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(f) \le 4g + 3 + (-1)^g$ for all $g \ge 2$.

(2) Let $f\colon N_{g,b} \to N_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(f) \le 2g - 1 + (-1)^g$ for all $g \ge 3$.

Special homeomorphisms on small genus surfaces

Lemma Let $f\colon F_{0,b} \to F_{0,b}$ be an orientation-preserving homeomorphism on a sphere with b boundary components. If there is at least three boundary components on $F_{0,b}$ which are invariant under f, then $L(f) \neq 0$, hence $\mathfrak{fr}(f) = 1$.

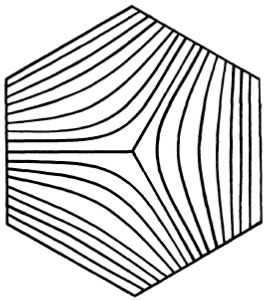
Lemma Let $f\colon F_{1,b} \to F_{1,b}$ be an orientation-preserving homeomorphism on a torus with $b \geq 1$ boundary components. If there is at least one boundary component of $F_{1,b}$ which is invariant under f, then $\mathfrak{fr}(f) \leq 6$.

Pseudo Anosov homeomorphisms

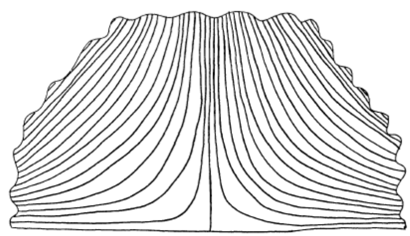
Lemma Let $\mathcal{F}$ be a singular foliation on a compact surface $F_{g,b}$. Then

$$\sum_{m=1}^{\infty}\Big((1-\frac{m}{2})Pr_m^{int}(\mathcal{F})-\frac{m}{2}Pr_m^{bd}(\mathcal{F})\Big)=\chi(F_{g,b})=2-2g-b,$$

where $Pr_m^{int}(\mathcal{F})$ is the number of m-prong singularities in the interior of $F_{g,b}$ and $Pr_m^{bd}(\mathcal{F})$ is the number of boundary components of $F_{g,b}$ with m-prong singularities.



A neighborhood of a 3-pronged singularity of a measured foliation



A neighborhood of a typical singularity of a measured foliation on ∂M

Lemma Let ψ be an orientation-preserving pseudo Anosov homeomorphism on a compact surface $F_{g,b}$ with $g \geq 2$. If the stable (and hence unstable) foliation of ψ has no 1-prong singularity, then there is a positive integer $n \leq 8g - 8$ such that $N(\psi^n) > 0$.

Proof.

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_g & \xrightarrow{\bar{\psi}} & F_g \end{array}$$

$\mathfrak{fr}^+(F_g) = 2g - 2$. $\bar{\psi}^{n_0}$ has a fixed point $\bar{x}_0$ with $n_0 \leq 2g - 2$. $ind(\bar{\psi}^{n_0}, \bar{x}_0) \neq 0$.

If $q^{-1}(\bar{x}_0)$ is a singleton, the point $q^{-1}(\bar{x}_0)$ is an isolated fixed point of ψ^{n_0} with non-zero index. $N(\psi^{n_0}) > 0$.

If $q^{-1}(\bar{x}_0)$ is a boundary component of $F_{g,b}$. Thus, $q^{-1}(\bar{x}_0)$ is an invariant circle of type (p_0, k_0) for ψ^{n_0}. $N(\psi^{p_0 n_0}) > 0$. It is sufficient to show that $p_0 n_0 \leq 8g - 8$. There are two cases:

Case (1): $p_0 = 2$ or 3. We have that $p_0 n_0 \leq 3(2g-2) = 6g-6$.

Case (2): $p_0 \geq 4$. We have

$$2 - 2g = \chi(F_g) = \sum_{p=1}^{\infty}(1 - \frac{p}{2})Pr_p^{int}(q(\mathcal{F}^s)) \leq (1 - \frac{p_0}{2})Pr_{p_0}^{int}(q(\mathcal{F}^s)),$$

because $q(\mathcal{F}^s)$ has no 1-prong singularity. It follows that

$$p_0 Pr_{p_0}^{int}(q(\mathcal{F}^s)) \leq (4g-4)(1 + \frac{2}{p_0 - 2}) \leq 8g - 8.$$

Since ψ permutes the boundaries of p_0-prong, we have that

$$n_0 \leq Pr_{p_0}^{bd}(\mathcal{F}^s) \leq Pr_{p_0}^{bd}(\mathcal{F}^s) + Pr_{p_0}^{int}(\mathcal{F}^s) = Pr_{p_0}^{int}(q(\mathcal{F}^s)),$$

and hence $p_0 n_0 \leq 8g - 8$. $\square$

Main results

Lemma Let $F_{g,b}$ be connected compact surface of genus g with b boundary components, where $g \geq 2$. Then for any orientation-preserving homeomorphism $\psi\colon F_{g,b} \to F_{g,b}$, there is a positive integer n with $n \leq 12g-12$ such that $N(\psi^n) > 0$.

Proof. The procedure of our proof will be fulfilled by using a reduction on the pairs (g, b) according to the lexicographic order. That is, we say $(g', b') < (g'', b'')$ if either $g' < g''$ or $g' = g''$ and $b' < b''$.

By the homotopy invariance of Nielsen number, we may assume that ψ is in standard form.

Case (1) $b = 0$. Done by S. Wang [7, Theorem 1].

Case (2) ψ is periodic. Done.

Case (3) ψ is a pseudo-Anosov map. Note that the homeomorphism ψ permutes the boundary components of type $(1,0)^+$. Let l_0 be the minimal length of orbits of ψ-action on the set of all boundary components of type $(1,0)^+$. We have three subcases according to the value of l_0.

Subcase (3.1): $l_0 = 0$. Done.

Subcase (3.2): $0 < l_0 \leq 12g - 12$. $N(\psi^{l_0}) > 0$.

Subcase (3.3): $l_0 > 12g - 12$. We collapse each boundary component of type $(1,0)^+$ to one point. We have a commutative diagram

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_{g,b'} & \xrightarrow{\bar{\psi}} & F_{g,b'} \end{array}$$

Of course, $\bar{\psi}$ is not in standard form. By definition of l_0, we have that $\mathrm{Fix}(\psi^m) = \mathrm{Fix}(\bar{\psi}^m)$ for any m with $0 < m \leq 12g - 12$. Any essential fixed point class of $\bar{\psi}^m$ contains at least one essential fixed point class of ψ^m if $m \leq 12g - 12$. Thus, it is sufficient to prove that $N(\bar{\psi}^m) > 0$ for some m with $m \leq 12g - 12$. This is just the inductive assumption.

Case (4) ψ is reducible. Let P_0 be a reduced piece with the biggest genus among all pieces. Assume that $P_0 \cong F_{g_0,b_0}$.

Thus, either $g_0 < g$ or $g_0 = g$ and $b_0 < b$. Note that ψ permutes all pieces.

Subcase (4.1): $g_0 \geq 2$. We write l_0 for the orbit length of P_0 under the action of ψ. That is $\psi^{l_0}(P_0) = P_0$, and $\psi^j(P_0) \neq P_0$ for $j = 1, 2, \ldots, l_0 - 1$. Clearly, $(\psi|_{P_0})^{l_0}$ is a homeomorphism on $P_0 \cong F_{g_0,b_0}$. By assumption of reduction, there is a positive number n_0 with $n_0 \leq 12g_0 - 12$ such that $N((\psi|_{P_0})^{l_0})^{n_0})) > 0$, i.e. $N(\psi^{l_0 n_0}|_{P_0}) > 0$. We have that $N(\psi^{l_0 n_0}) > 0$. Clearly,

$$l_0 n_0 \leq l_0(12g_0 - 12) \leq 12g - 12l_0 \leq 12g - 12.$$

Subcase (4.2): $g_0 = 0$ or 1. We have a careful analysis like the above subcase combined with the two lemma on small genus surfaces. $\square$

Theorem

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

Proof. Consider a homeomorphism $\psi\colon F_{g,b} \to F_{g,b}$, where $g \geq 2$. Then ψ^2 must be orientation preserving.

Let $\eta\colon F_{g-1,2b} \to N_{g,b}$ be the classical orientation covering. Write τ for the unique non-trivial covering transformation. Any homeomorphism $\psi\colon N_{g,b} \to N_{g,b}$ has two liftings ϕ and $\tau\phi$. Without loss of the generality, we may assume that ϕ is orientation preserving. There is a positive integer n with $n \leq 12(g-1) - 12$ such that ϕ^n has a fixed point x_0. Clearly, $\psi^n(\eta(x_0)) = \eta(\phi^n(x_0)) = \eta(x_0)$, i.e. $\eta(x_0)$ is a fixed point of ψ^n. This implies that $\mathfrak{fr}(N_{g,b}) \leq 12(g-1) - 12 = 12g - 24$. $\square$

THANK YOU!

References

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