Concordance class of the Hopf link

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Outline

- Concordance
- Main result
- Covering link calculus
- More results
- Related work

Concordance

- Let L_0 and L_1 be ordered, oriented, n-component links.

Definition L_0 is smoothly (resp. topologically) concordant to L_1 if there is a smooth (resp. topologically locally flat) embedding

$$h\colon \coprod_n S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$$

with $h(\coprod_n S^1 \times i) = L_i,\ i = 0, 1.$

- Smooth concordance $\Rightarrow$ topological concordance
- A link concordant to the unlink is called slice.

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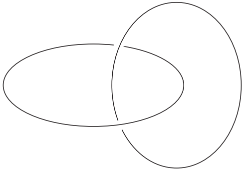
- $\mathcal{C}^{smooth} := (\{\text{knots}\}/\text{smooth concordance}, \#)$, the smooth knot concordance group
- $\mathcal{C}^{top} := (\{\text{knots}\}/\text{top. concordance}, \#)$, the topological knot concordance group

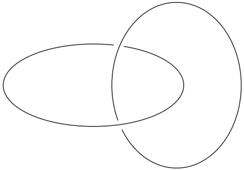
Corollary The natural surjection $\mathcal{C}^{smooth} \to \mathcal{C}^{top}$ is not injective.

Concordance: 2-component link



- Let HL be the Hopf link. Then the Alexander polynomial $\Delta_{HL}(t_1, t_2) = 1$.

Concordance: 2-component link



- Let HL be the Hopf link. Then the Alexander polynomial $\Delta_{HL}(t_1, t_2) = 1$.

Theorem (J. F. Davis, '2004) Let L be a 2-component link. If $\Delta_L(t_1, t_2) = 1$, then L is topologically concordant to the Hopf link.

Concordance: 2-component link

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Answer: Yes. Let $HL = (L_1, L_2)$ be the Hopf link and let $\mathrm{Wh}(T)$ be the positive Whitehead double of the right-handed trefoil. Take $L = (L_1 \# \mathrm{Wh}(T), L_2)$.

Main Theorem

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Main Theorem (Cha-K.-Ruberman-Strle, '2010) There exists a 2-component link L satisfying the following:

1. All components of L are unknotted.
2. $\Delta_L(t_1, t_2) = 1$.
3. L is not smoothly concordant to the Hopf link.

Covering link calculus

Let p be a prime and $L = (L_1, L_2)$ a 2-component link in S^3. Let Y be the p^a-fold cyclic cover of S^3 branched along L_2.

Definition The preimage of L_1 in Y is called a covering knot of L.

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Lemma If L and L' are concordant 2-component links in S^3, then their p^a-fold covering knots J and J' are rationally concordant, that is, concordant in a rational homology $S^3 \times [0, 1]$.

Corollay Suppose L is smoothly concordant to the Hopf link. Then the p^a-fold covering knot of L is smoothly rationally concordant to the unknot.

Smooth concordance invariant from knot Floer homology

Theorem (Ozsváth-Szabó) There is a $\mathbb{Z}$-valued knot invariant τ satisfying the following:

1. $\tau(K) = 0$ if K is smoothly rationally concordant to the unknot,
2. $\tau(K \# J) = \tau(K) + \tau(J)$,
3. $\tau(-K) = -\tau(K)$,
4. $\tau(K) = \tau(K^r)$ where K^r the orientation reverse of K,
5. $\tau(K) \leq |g_4(K)|$ where $g_4(K)$ is the smooth 4-ball genus of K.
6. $\tau(K) = -\sigma(K)/2$ if K is alternating.

Smooth concordance invariant from knot Floer homology I

Corollary The τ-invariant induces a homomorphism $\tau\colon \mathcal{C}^{smooth} \to \mathbb{Z}$.

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Corollary The τ-invariant induces a homomorphism $\tau \colon \mathcal{C}^{smooth} \to \mathbb{Z}$.

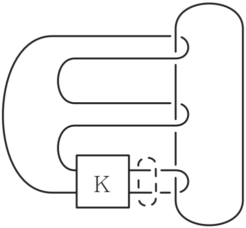
Theorem (Manolescu-Owens, Hedden) If $\tau(K) > 0$, then the Whitehead double $\mathrm{Wh}(K)$ is not smoothly rationally concordant to the unknot. In fact, $\tau(\mathrm{Wh}(K)) = 1$.

Proof of Main Theorem

Key observation:

- $L = (L_1, L_2)$: 2-component link with L_2 unknotted.
- Then the 2-fold cover of S^3 branched along L_2 is S^3.
- $\widetilde{L_1}$:= covering knot of L (in S^3).
- If $\tau(\widetilde{L_1}) \neq 0$, then L is not smoothly concordant to the Hopf link.

Proof of Main Theorem



- Let $L(K)$ be the link above where K is a knot with $\tau(K) > 0$ (cf. $K =$ the right-handed trefoil).

Proof of Main Theorem

- The components of $L(K)$ are unknotted and $\Delta_{L(K)}(t_1, t_2) = 1$.
- The double cover of $L(K)$ branched along the right component of $L(K)$ is S^3.
- The covering knot, which is the preimage of the left component of $L(K)$, is $\mathrm{Wh}(K \# K^r)$.

Proof of Main Theorem

- The components of $L(K)$ are unknotted and $\Delta_{L(K)}(t_1, t_2) = 1$.
- The double cover of $L(K)$ branched along the right component of $L(K)$ is S^3.
- The covering knot, which is the preimage of the left component of $L(K)$, is $\mathrm{Wh}(K\#K^r)$.
- $\tau(K\#K^r) = \tau(K) + \tau(K^r) = \tau(K) + \tau(K) = 2\tau(K) > 0$.
- $\tau(\mathrm{Wh}(K\#K^r)) = 1$. Therefore $L(K)$ is not smoothly concordant to the Hopf link.

Infinitely many smooth concordance classes

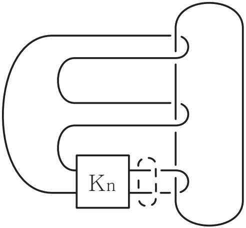
Theorem (Cha-K.-Ruberman-Strle, '2010) There are 2-component links $L(n)$ ($n \in \mathbb{N}$) such that $L(n)$ has Alexander polynomial 1 and unknotted components and $L(n)$ are mutually distinct up to smooth concordance.

Blow-down for links

- Let $L = (L_1, L_2)$ is a 2-component link with L_1 unknotted.
- ± 1 surgery on L_1 in S^3 is again S^3.
- The image of L_2, say $\check{L}$, is a knot in S^3
- We say that $\check{L}$ is obtained by blowing down L_1.

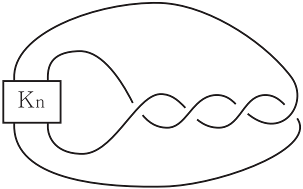
Lemma Suppose two 2-component links L and L' are smoothly concordant, and that L_1 and L'_1 are both unknotted. Then the knots $\check{L}$ and $\check{L}'$ are smoothly concordant in a homotopy $S^3 \times [0, 1]$.

Proof of Theorem: examples



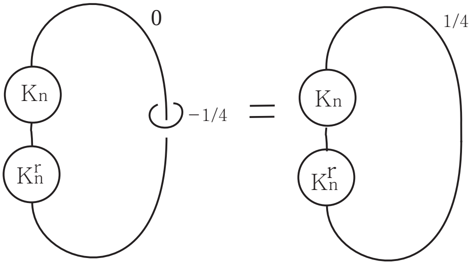
- Let K_n be the $(2, 2n+1)$ torus knot and $L(n) := L(K_n)$.

Proof of Theorem: blow-down



- By blowing down the right component of $L(n)$, we obtain the above knot $\check{L}(n)$ in S^3.
- Suppose $L(m)$ and $L(n)$ are smoothly concordant. Then $\check{L}(m)$ and $\check{L}(n)$ are smoothly concordant in a homotopy $S^3 \times [0,1]$.

Proof of Theorem: double branched cover



- Then $M_2(\check{L}(m))$ and $M_2(\check{L}(n))$, the double covers of S^3 branched along $\check{L}(m)$ and $\check{L}(n)$, are $\mathbb{Z}_2$-homology cobordant.
- $M_2(\check{L}(m))$ is diffeomorphic to $S^3_{1/4}(K_m \# K_m^r)$, $m \in \mathbb{N}$.

Proof of Theorem: d-invariant (or the correction term)

Theorem (Ozsváth-Szabó) If two rational homology 3-spheres M and M' are smoothly $\mathbb{Z}_2$-homology cobordant, then $d(M) = d(M')$.

Proof of Theorem: d-invariant (or the correction term)

Theorem (Ozsváth-Szabó) If two rational homology 3-spheres M and M' are smoothly $\mathbb{Z}_2$-homology cobordant, then $d(M) = d(M')$.

- $d(S^3_{1/4}(K_m \# K_m^r)) = d(S^3_1(K_m \# K_m^r)) = -2m$.
- Therefore $d(S^3_{1/4}(K_m \# K_m^r)) \neq d(S^3_{1/4}(K_n \# K_n^r))$, which is a contradiction.

Related work

Theorem (Cha-Ruberman, '2010) There are topologically slice links that have smoothly slice components but are not smoothly concordant to any link with unknotted components.

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- Concordance between links with nontrivial linking numbers: Stefan Friedl-Mark Powell, Jae Choon Cha, Min Hoon Kim, ...

Related work

Let $\mathcal{C}_T := \text{Ker}\{\mathcal{C}^{smooth} \to \mathcal{C}^{top}\}$ and $\mathcal{C}_\Delta :=$ the subgroup of $\mathcal{C}^{smooth}$ generated by knots with Alexander polynomial 1.

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Theorem (Hedden-Livingston-Ruberman) The quotient group $\mathcal{C}_T/\mathcal{C}_\Delta$ is infinitely generated.

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- Concordance between knots with coprime Alexander polynomials.