On connected sum of

generalized Kinoshita's theta-curves

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Abstract

Kinoshita's theta-curve $\theta(1,1,1)$ is locally unknotted theta-curve, that is, its constituent knots are all trivial. We obtain generalized Kinoshita's theta-curve $\theta(i,j,k)$ by adding full-twists to $\theta(1,1,1)$. In this talk we discuss about order-3 vertex connected sums of $\theta(i,j,k)$ and $\theta(i',j',k')$.

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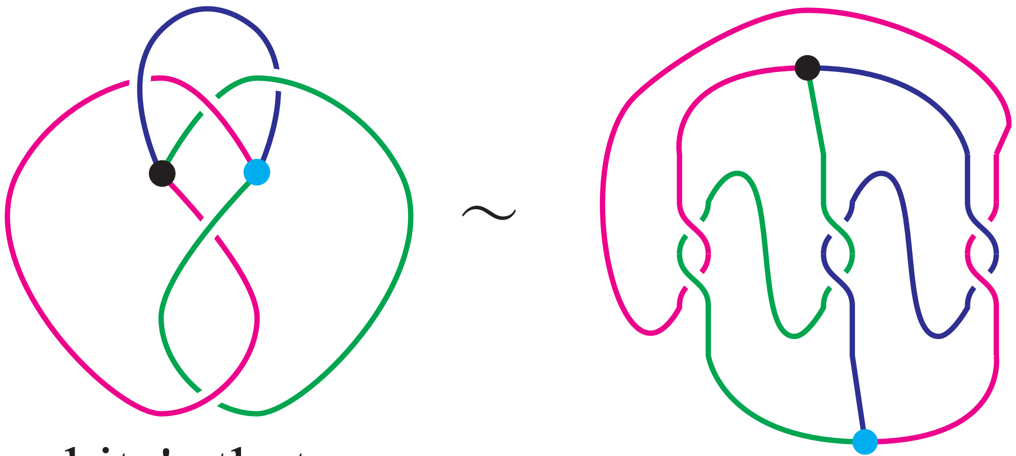
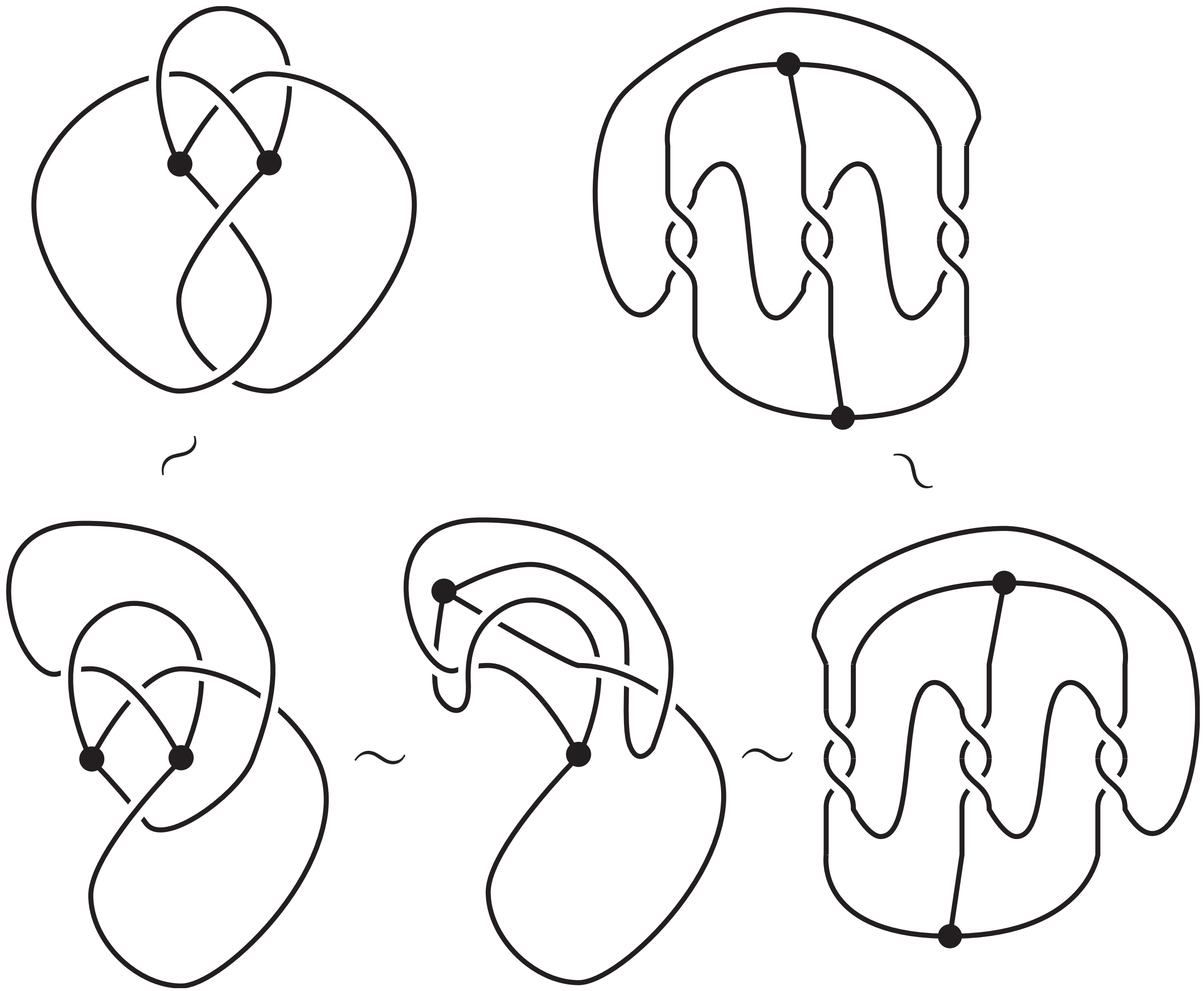


Kinoshita's theta-curve $\quad\quad$ $\theta(1,1,1)$



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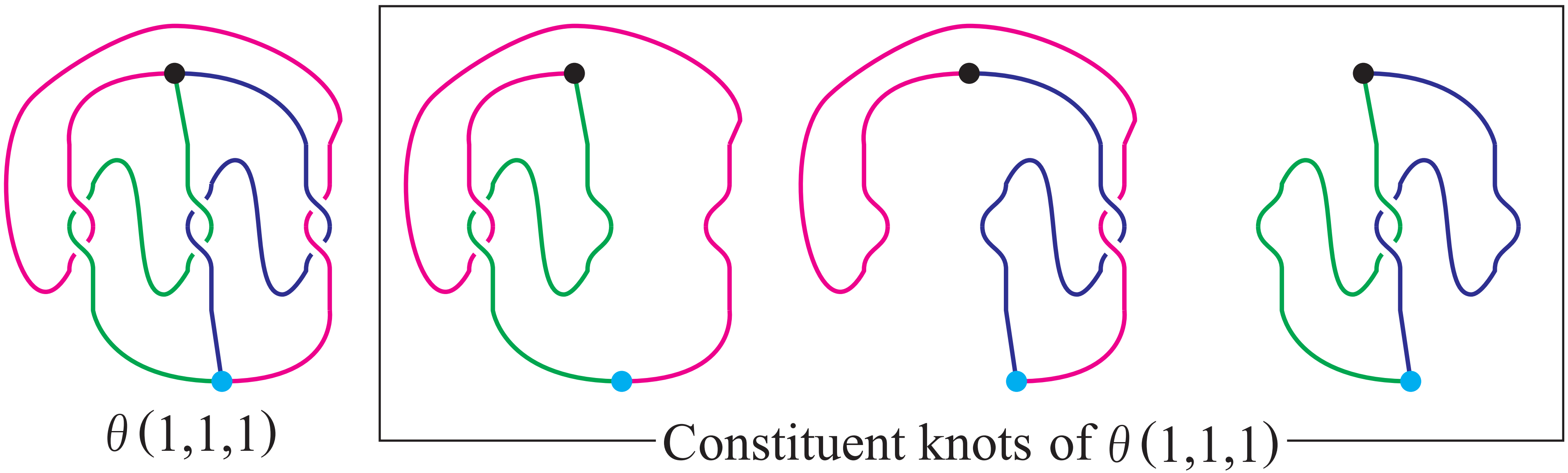

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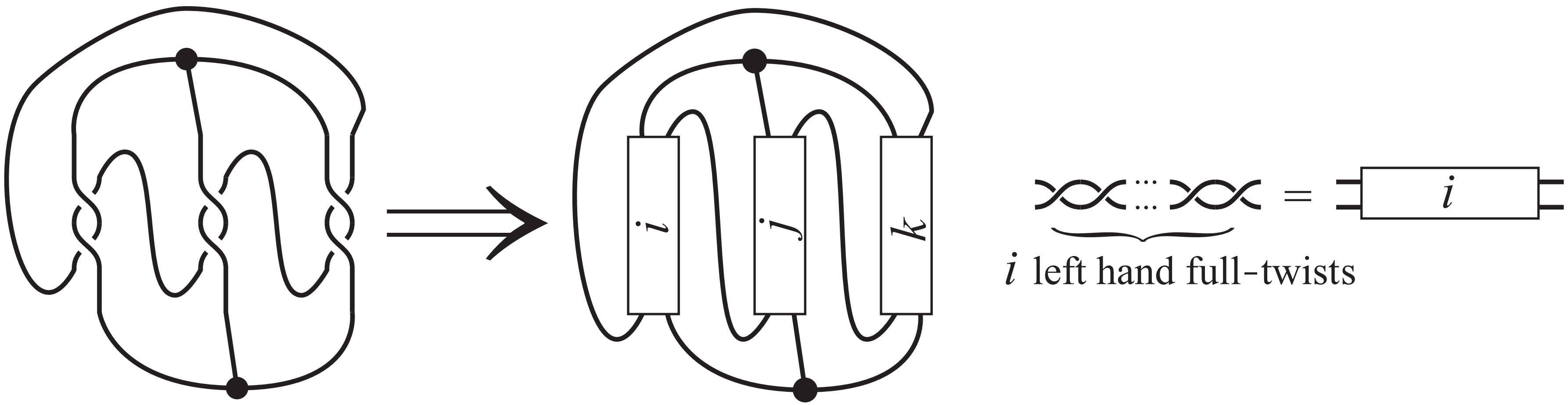

$\theta(1,1,1)$ Generalized Kinoshita's theta-curve $\theta(i,j,k)$

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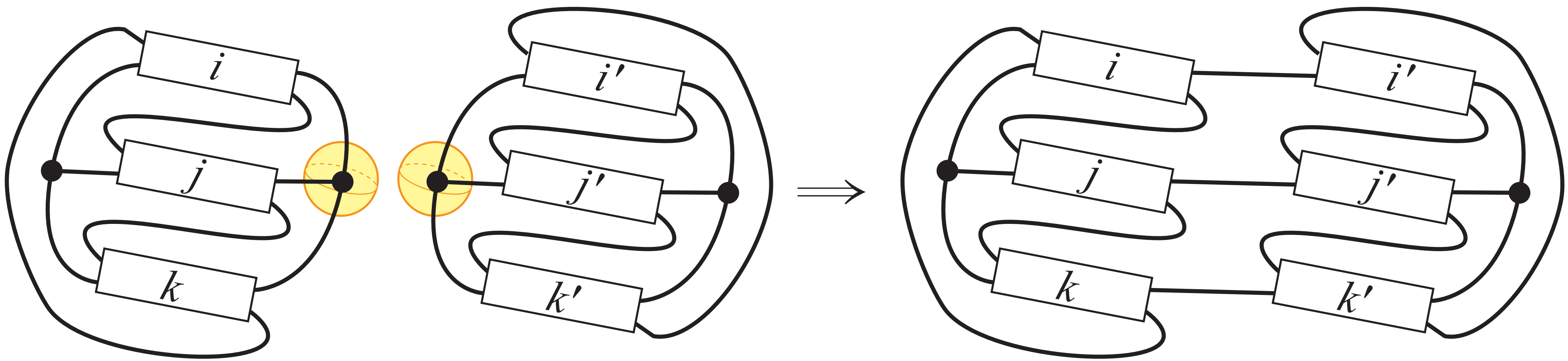
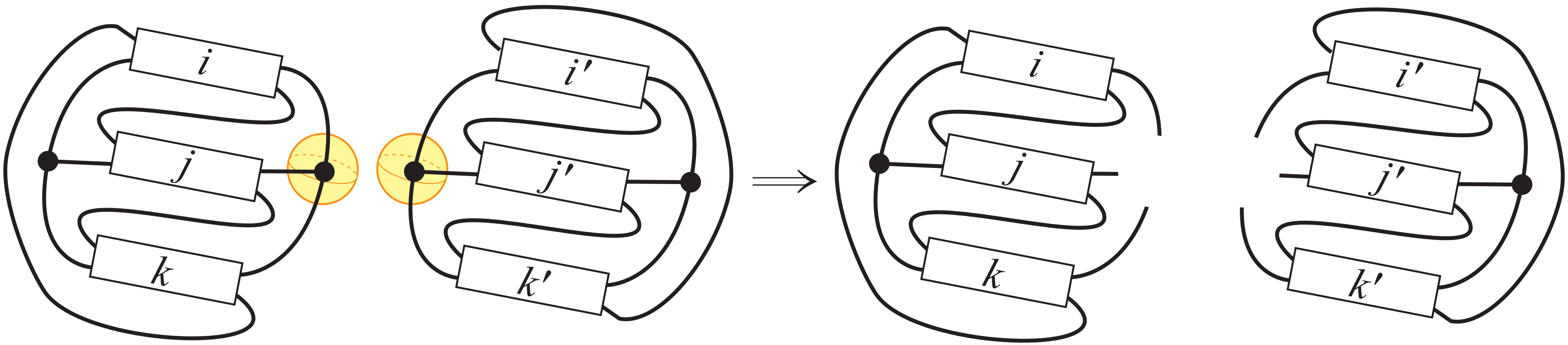


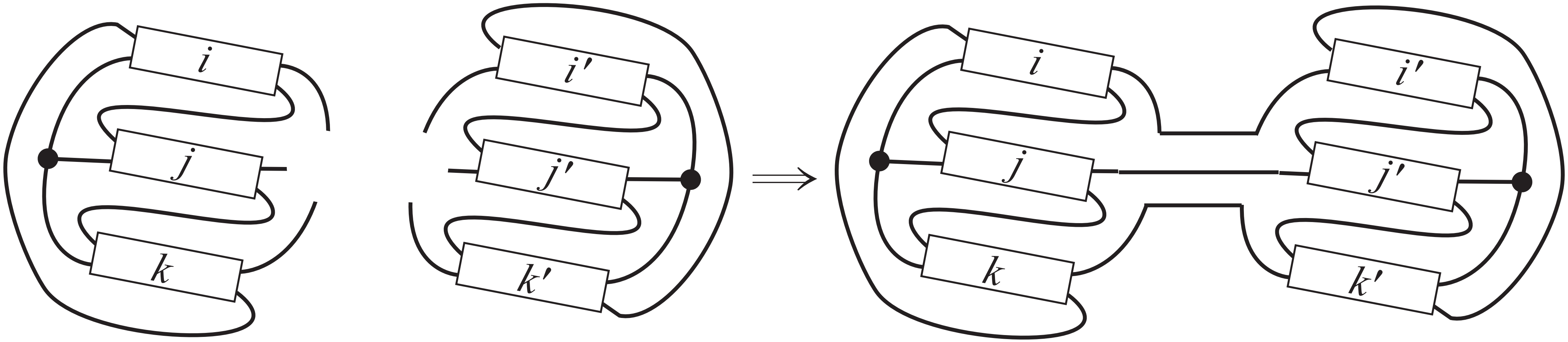
Order-3 vertex connected sum of $\theta(i,j,k)$ and $\theta(i',j',k')$

Contents

Order-3 vertex connected sum [Wolcott,1989]

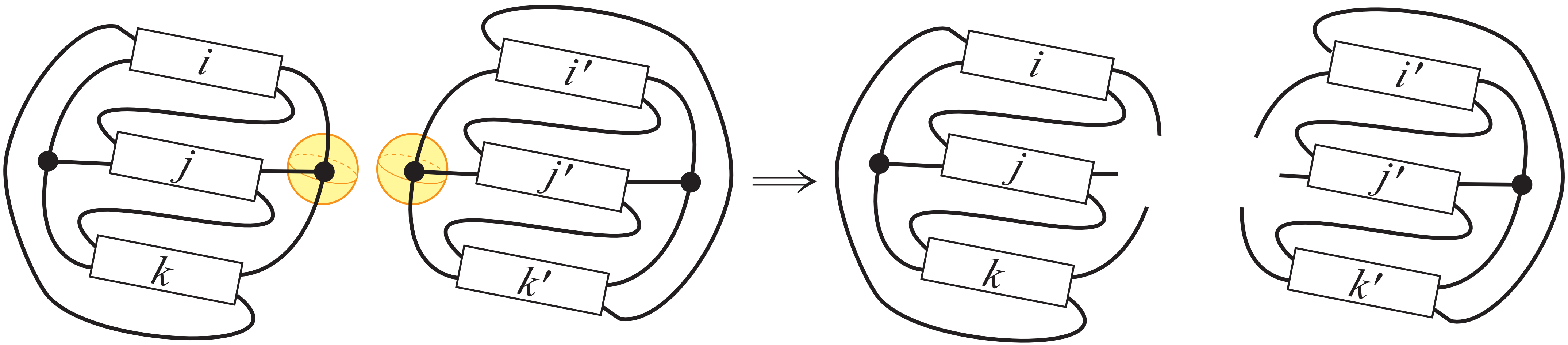


Remove 3-balls.



Glue the remaining 3-balls together.

Order-3 vertex connected sum [Wolcott, 1989]



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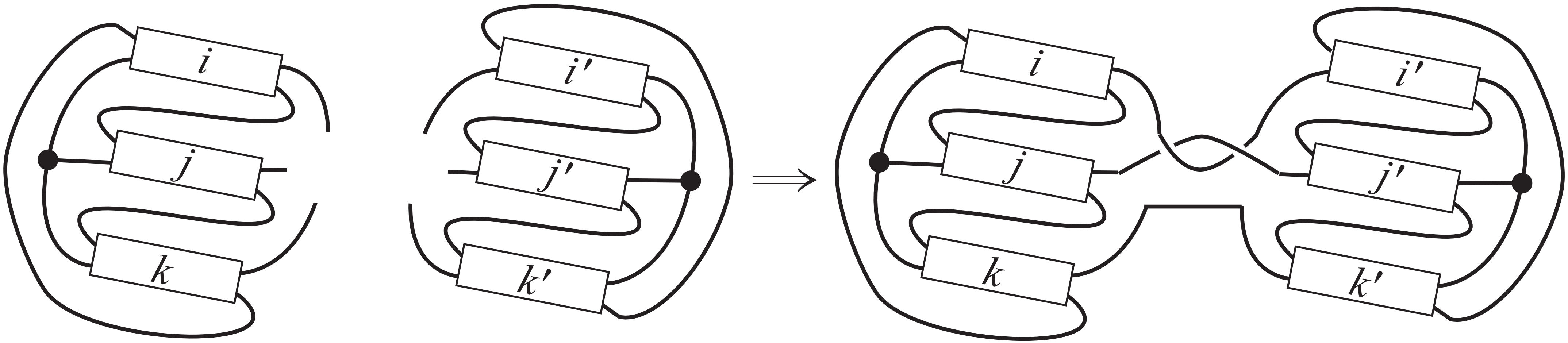
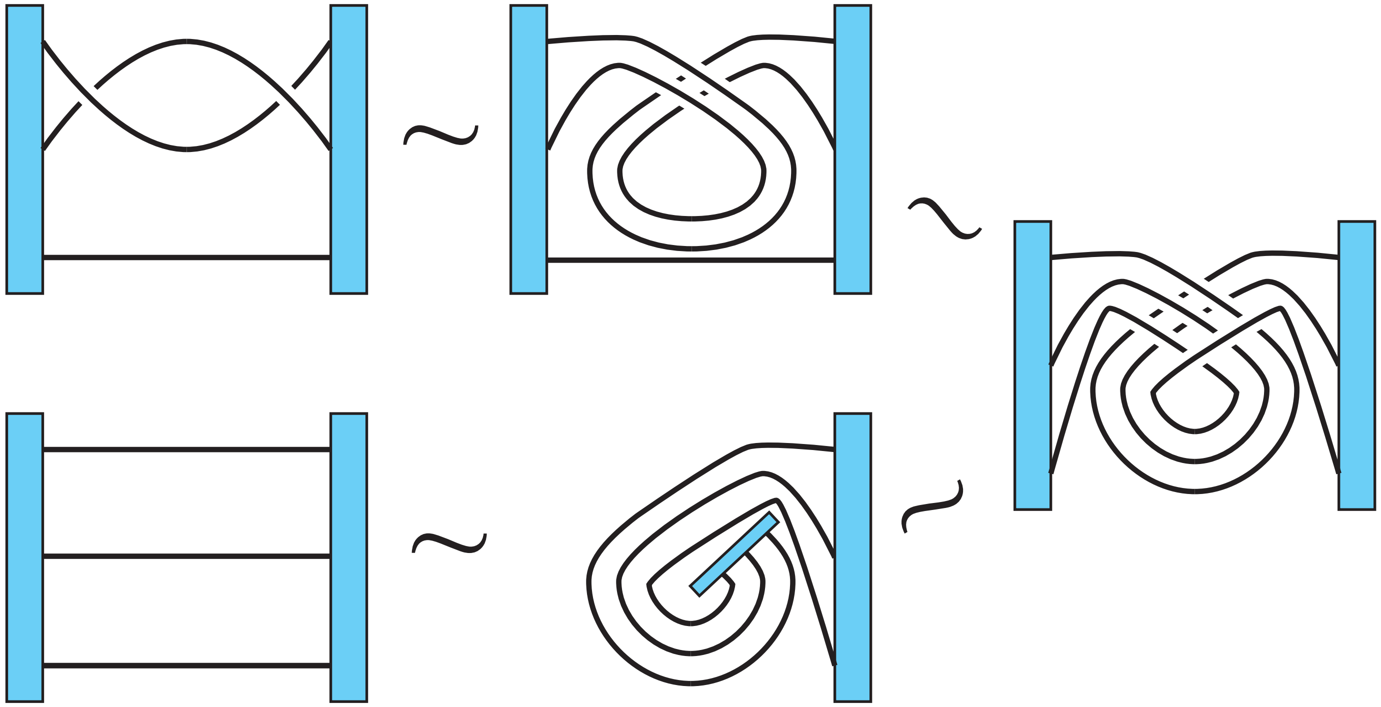
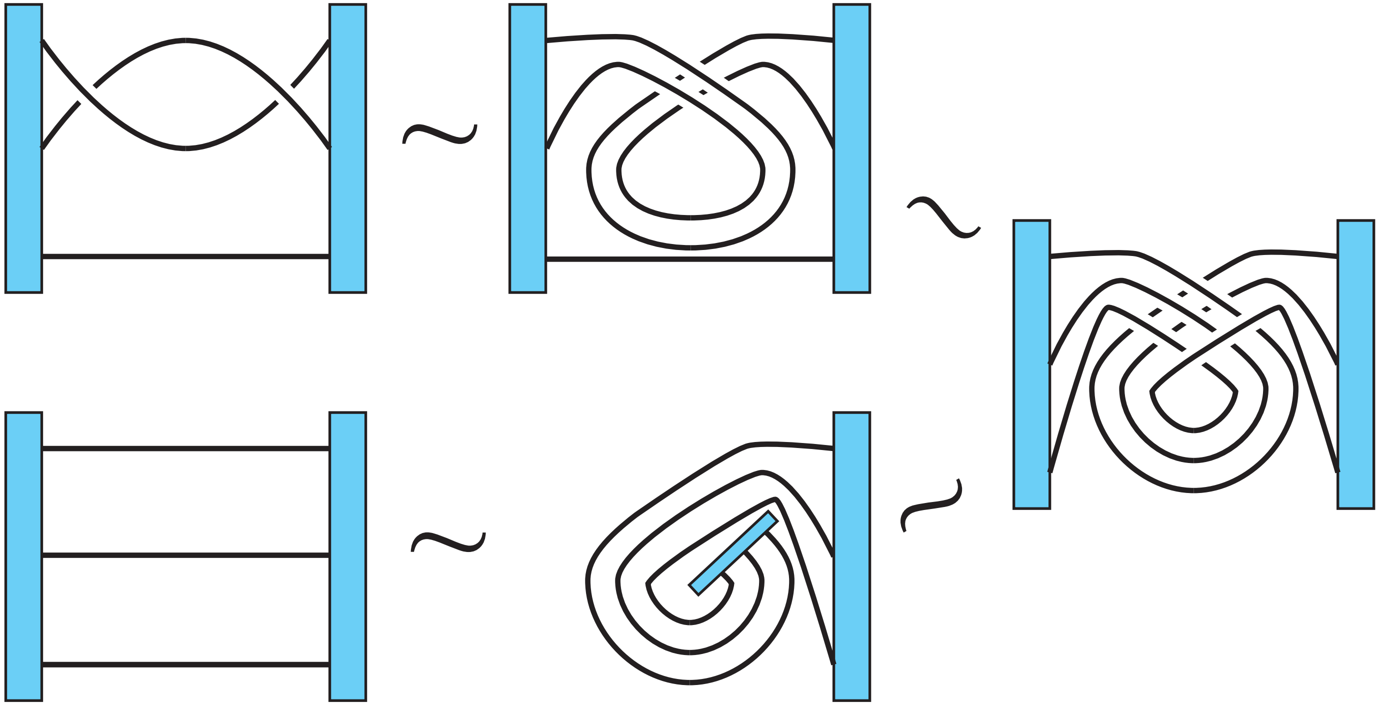
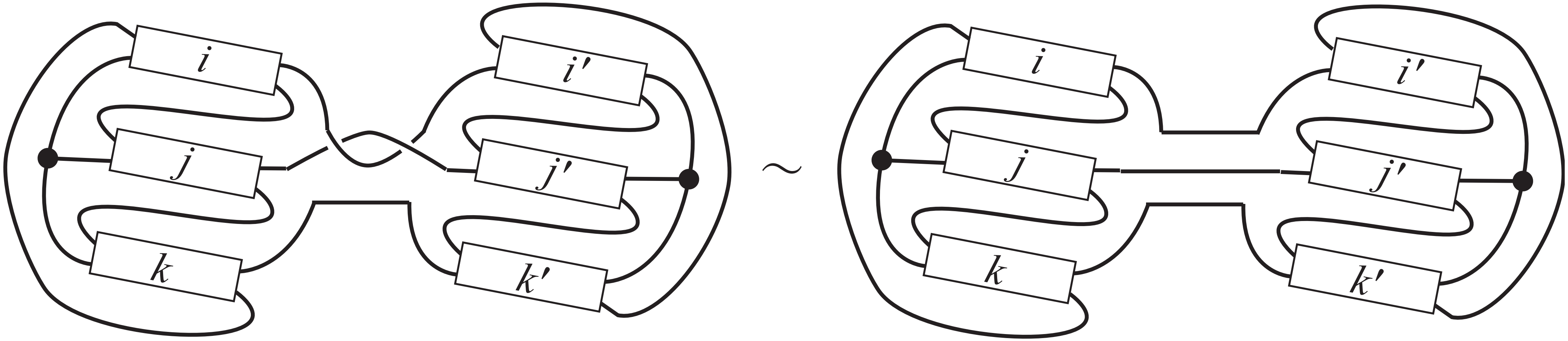


How about this situation?

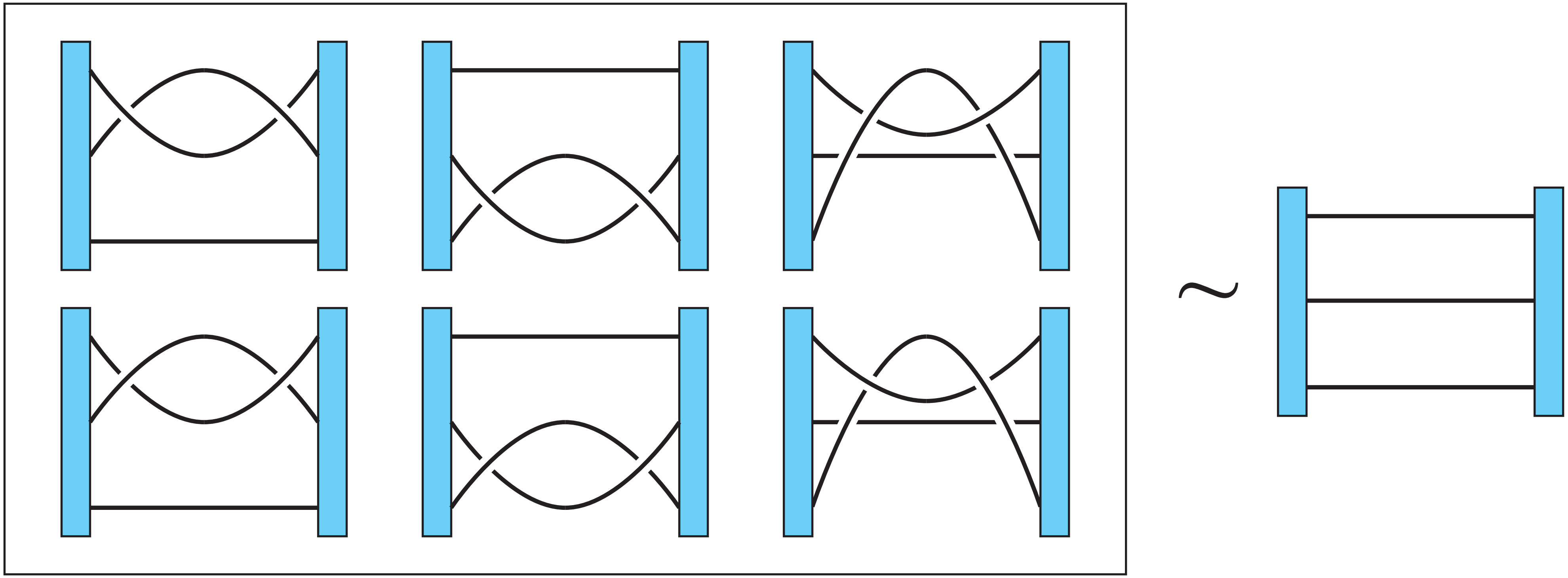




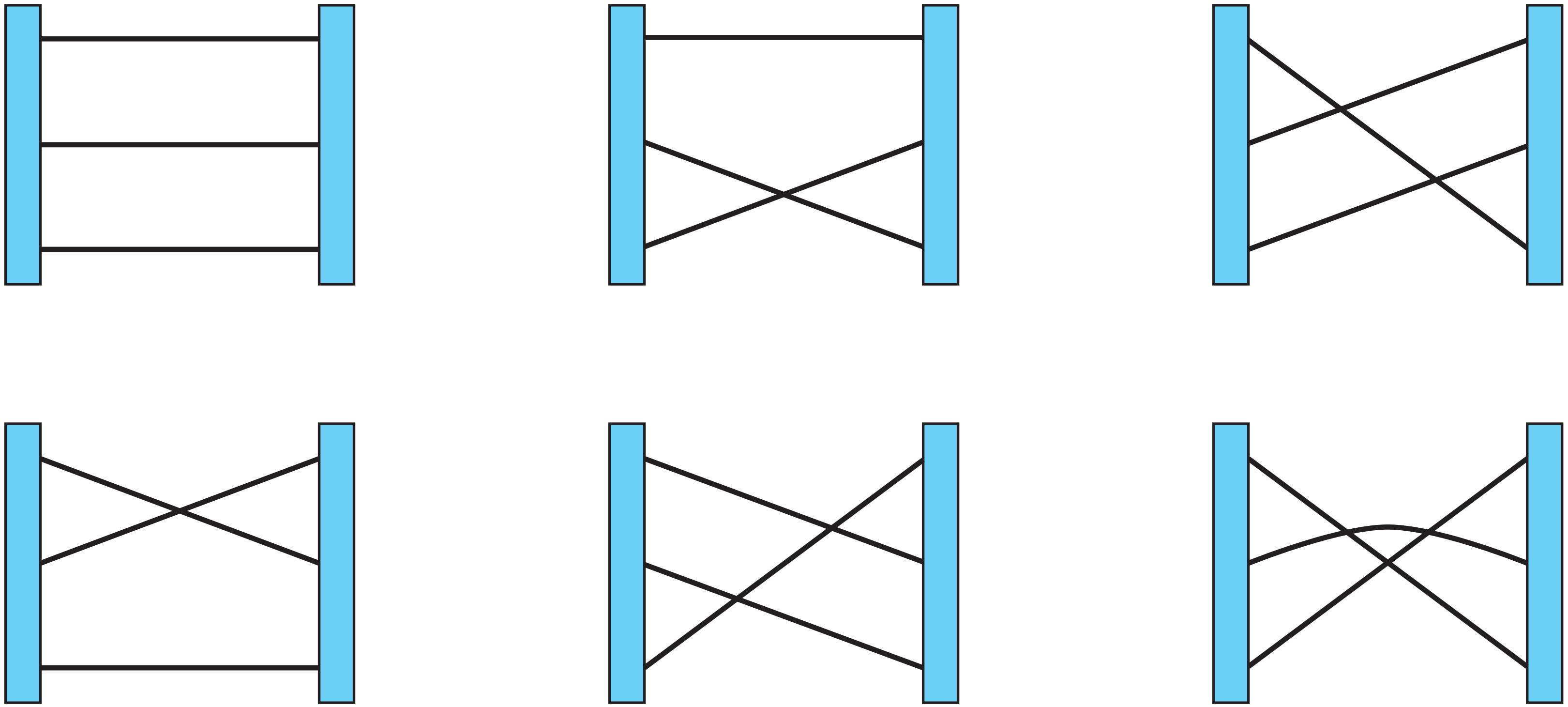


i
j
k
i′
j′
k′
i
j
k
i′
j′
k′

Full-twists of bonds can be canceled.

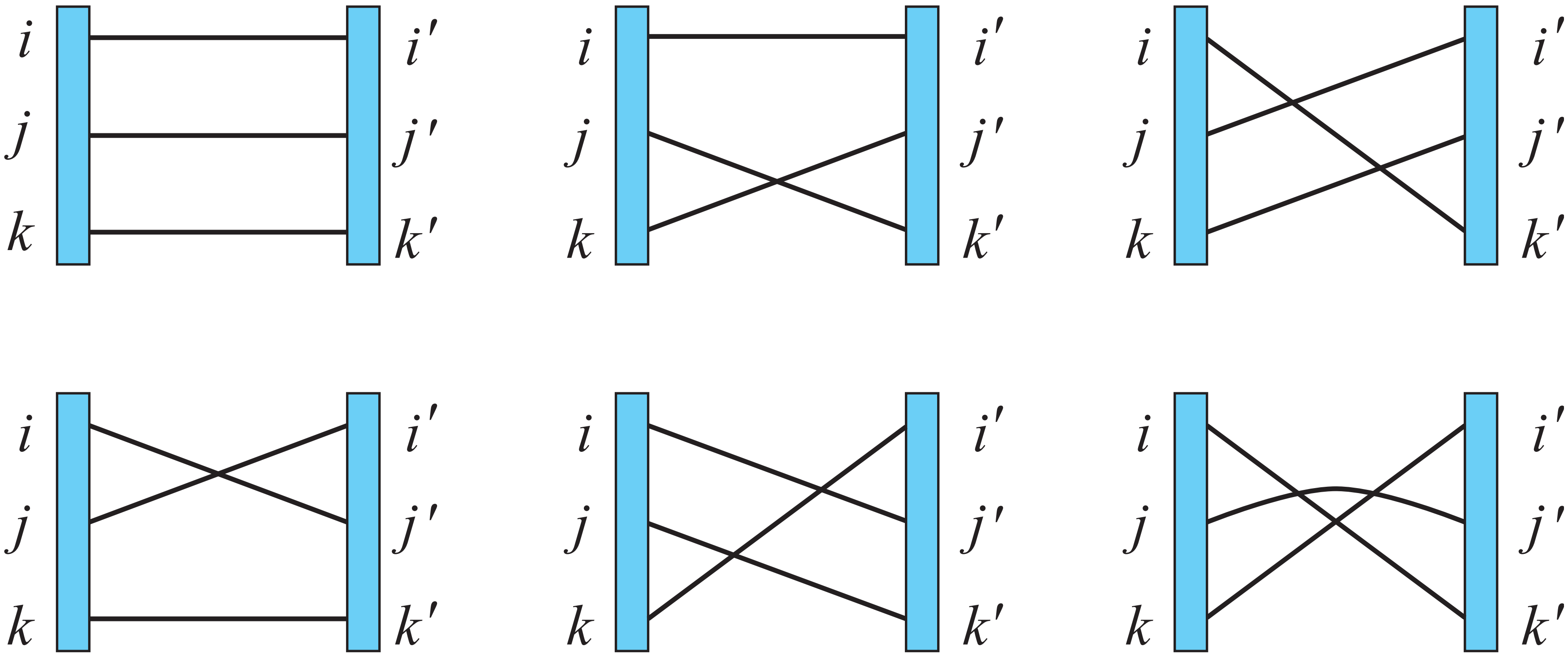


Fact. For any two given vertices, we need to consider only six different order-3 vertex connected sums.



Note. Over/under information of crossings are not important.

Fact. For any two given vertices, we need to consider only six different order-3 vertex connected sums.



$\theta(i,j,k) \#_3 \theta(i',j',k')$ $\quad$ $\theta(i,j,k) \#_3 \theta(i',k',j')$ $\quad$ $\theta(i,j,k) \#_3 \theta(k',i',j')$

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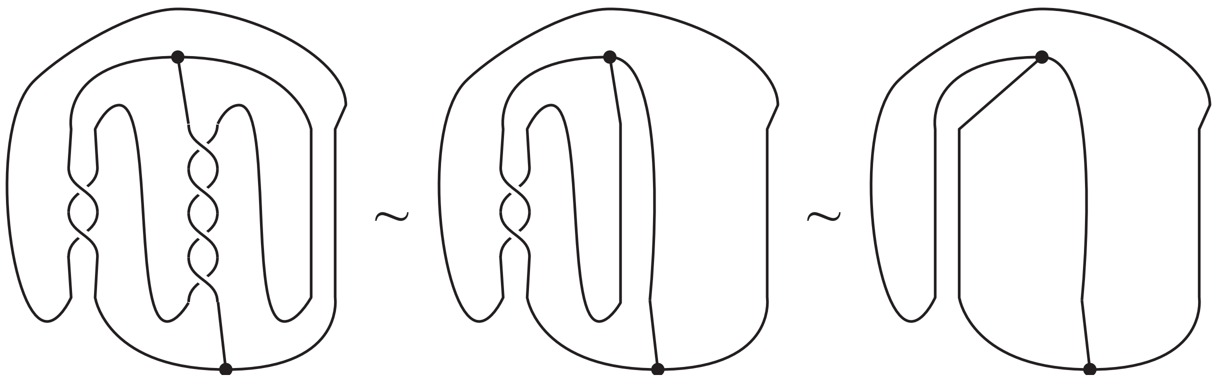
Generalized Kinoshita's theta-curve

Remark. If $ijk = 0$, generalized Kinoshita's theta-curve $\theta(i, j, k)$ is trivial.

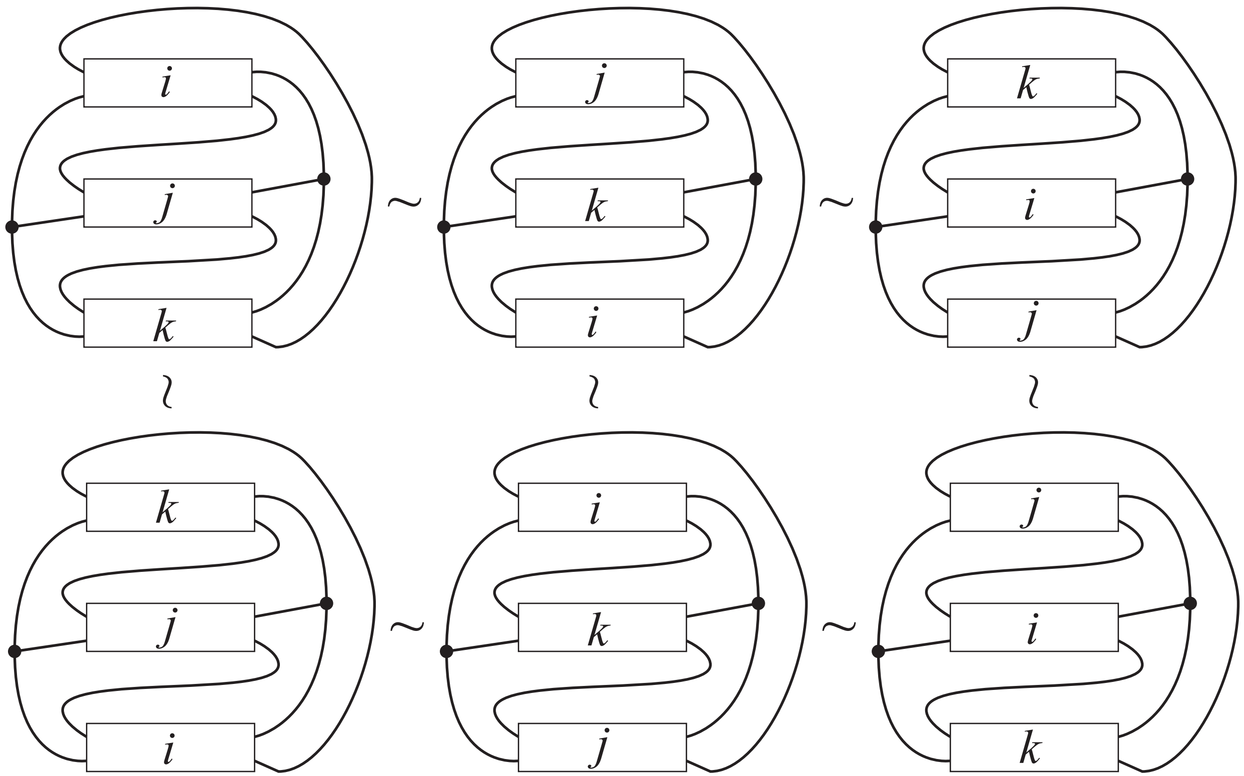
$(i = 1, j = 2, k = 0)$



From now on, we consider the case $ijk \neq 0$.

Remark. Generalized Kinoshita's theta-curve has symmetry.

That is, $\theta(i,j,k) \sim \theta(j,k,i) \sim \theta(k,i,j)$. Moreover, if we do not consider the orientation, $\theta(i,j,k) \sim \theta(j,k,i) \sim \theta(k,i,j) \sim \theta(j,i,k) \sim \theta(i,k,j) \sim \theta(k,j,i)$.



Question. $\theta(1,2,3)\,\#_3\,\theta(4,5,6) \overset{?}{\sim} \theta(1,2,3)\,\#_3\,\theta(5,4,6)$.

Remark. We can not detect the difference between $\theta(i,j,k)\,\#_3\,\theta(i',j',k')$ and $\theta(i,j,k)\,\#_3\,\theta(j',k',i')$ by the Yamada polynomial.

Theorem. [Yamada, 1989]

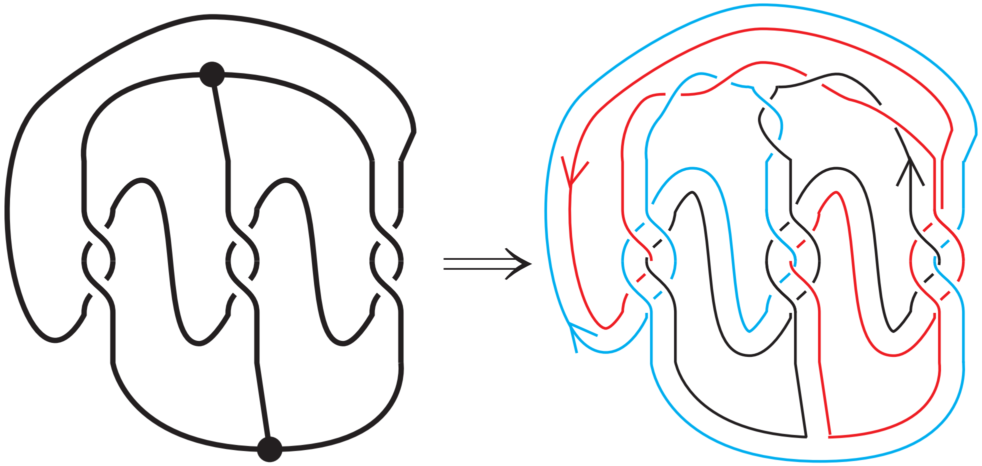
g, g': spatial graph diagrams

$R(g) \in \mathbf{Z}[A, A^{-1}]$: the Yamada polynomial of g

$$R(g\,\#_3\,g') = \frac{R(g)\,R(g')}{-A^2 - A - 2 - A^{-1} - A^{-2}} = \frac{R(g)\,R(g')}{R\left(\ominus\right)}$$

Therefore, we need another invariant.

Associated link [Kauffman-Simon-Wolcott-Zhao,1993]



$\mathrm{lk}(K_1, K_2) = 0$
$\mathrm{lk}(K_1, K_3) = 0$
$\mathrm{lk}(K_2, K_3) = 0$

Theorem. [Kauffman-Simon-Wolcott-Zhao,1993]

A theta-curve can be associated uniquely with a link of self linking number zero. Moreover, the link is an ambient isotopy invariant of the theta-curve.

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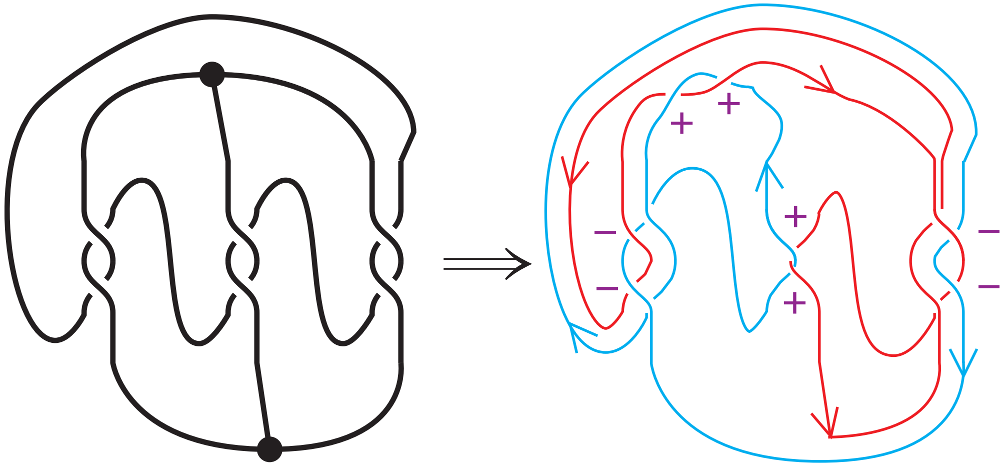

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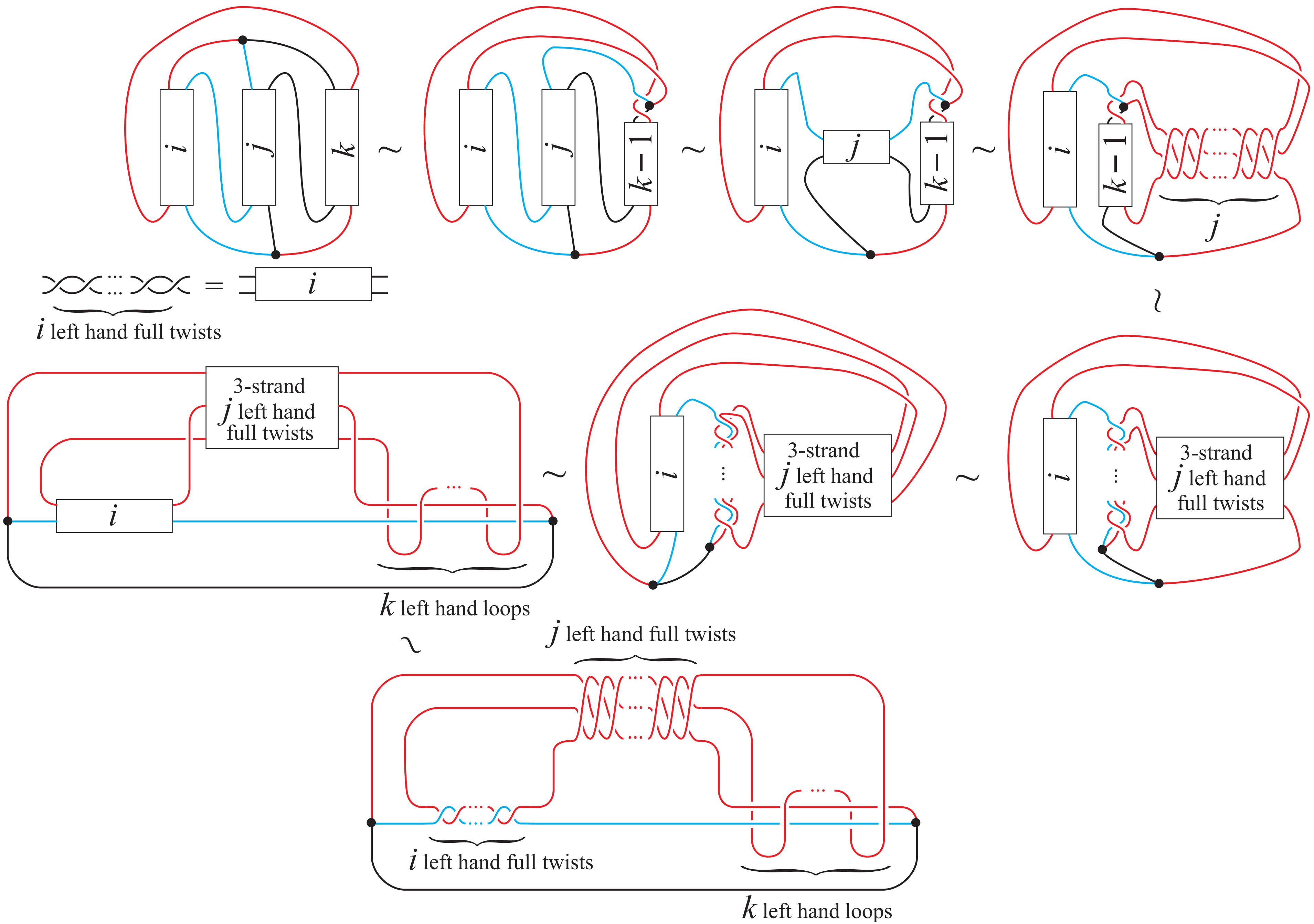
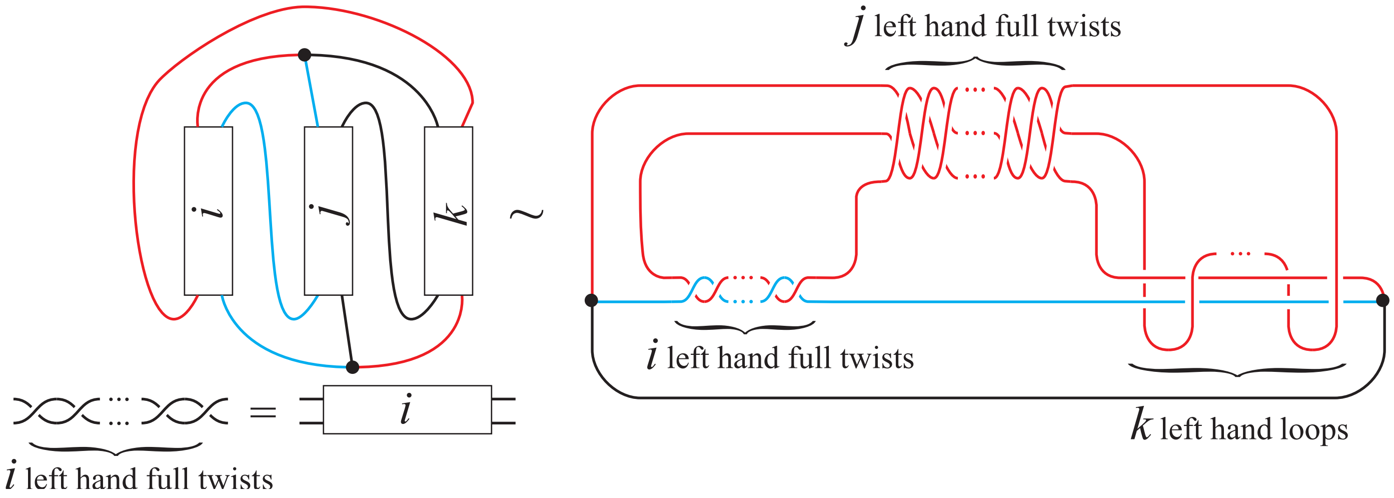
Theorem. [Kauffman-Simon-Wolcott-Zhao,1993]

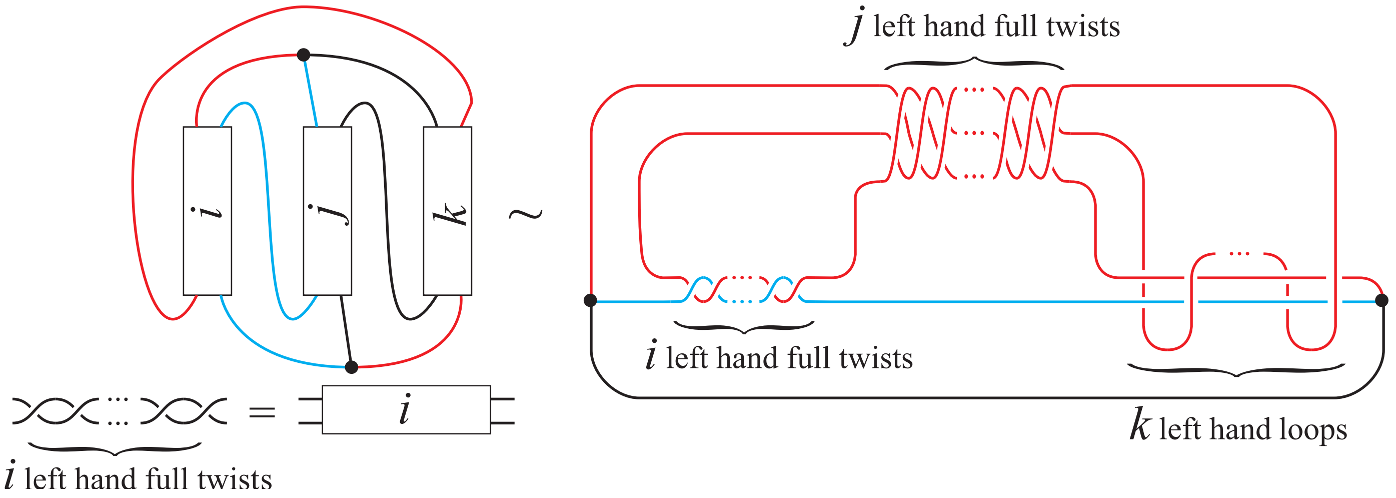
A theta-curve can be associated uniquely with a link of self linking number zero. Moreover, the link is an ambient isotopy invariant of the theta-curve.

First, we deform $\theta(i,j,k)$ into the following.

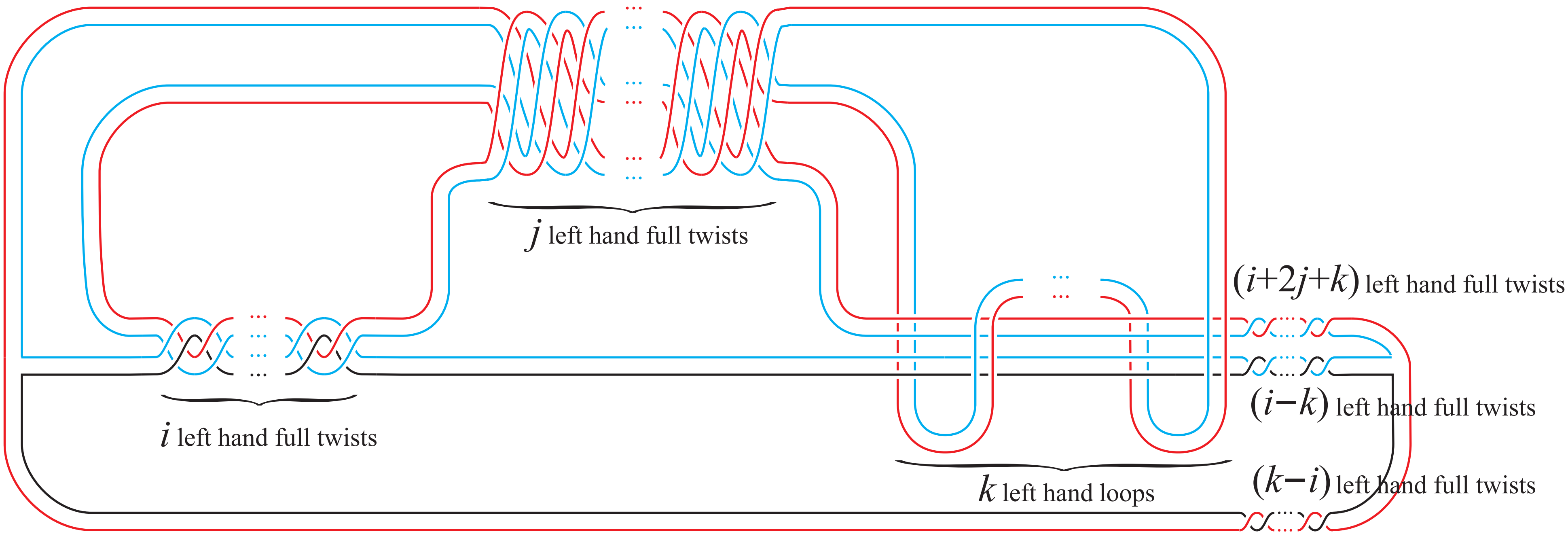




For $\theta(i, j, k)$, we construct the associated link.

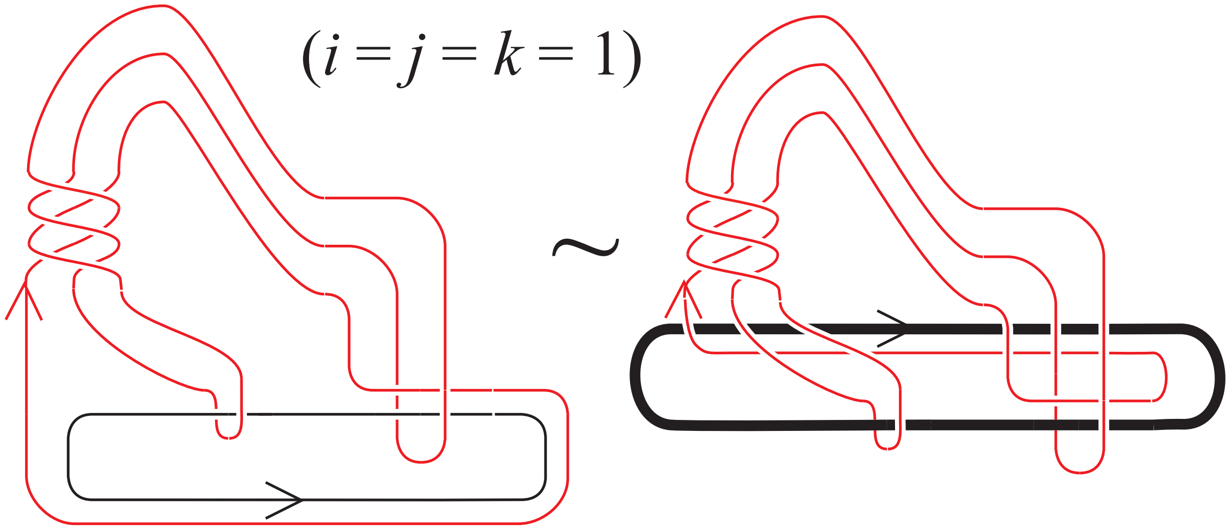


For $\theta(i,j,k)$, we construct the associated link.



Remark. The associated link of $\theta(i,j,k)$ consists of three trivial components.

Next, we consider 2-component sublink of the associated link.



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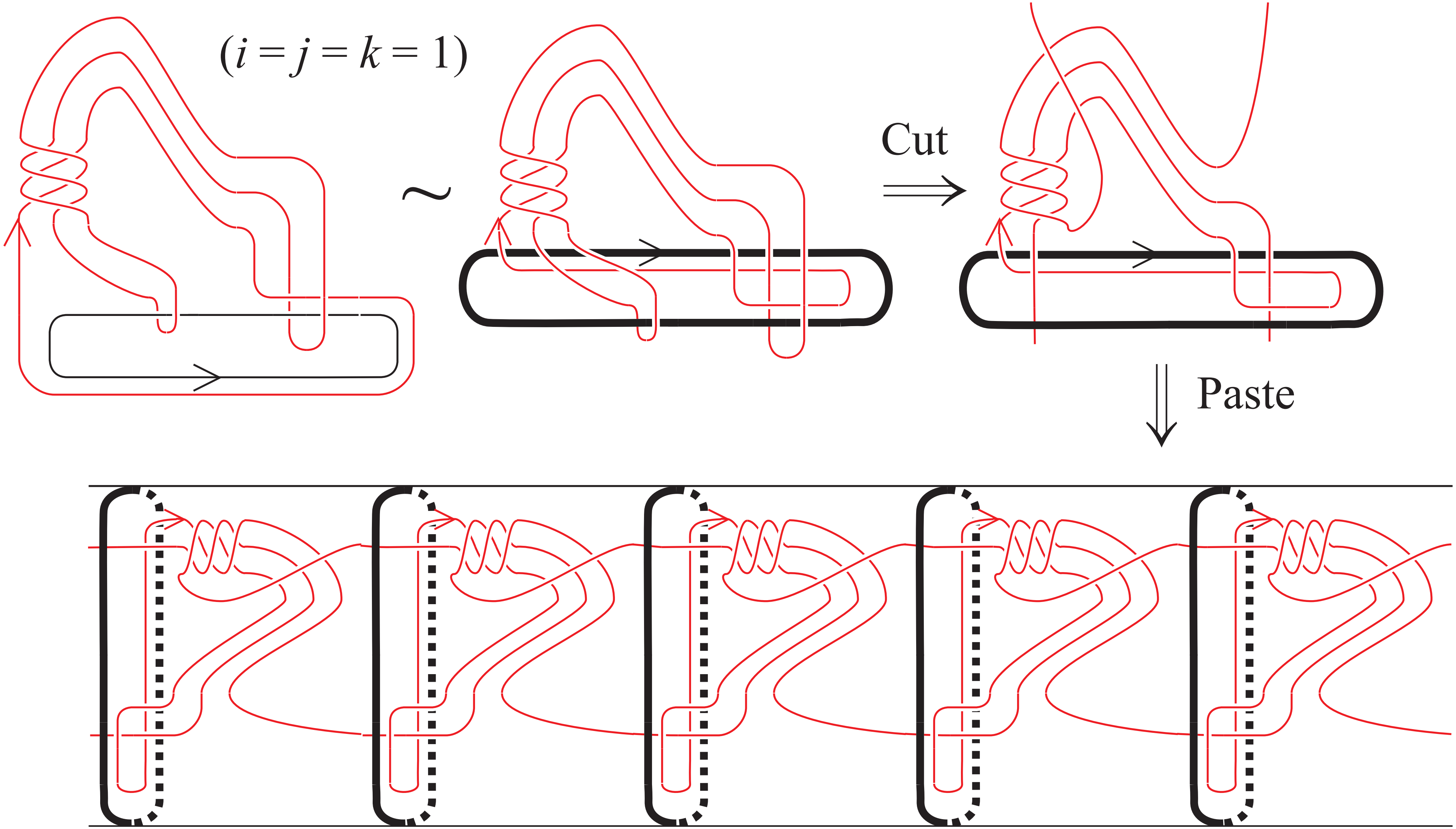
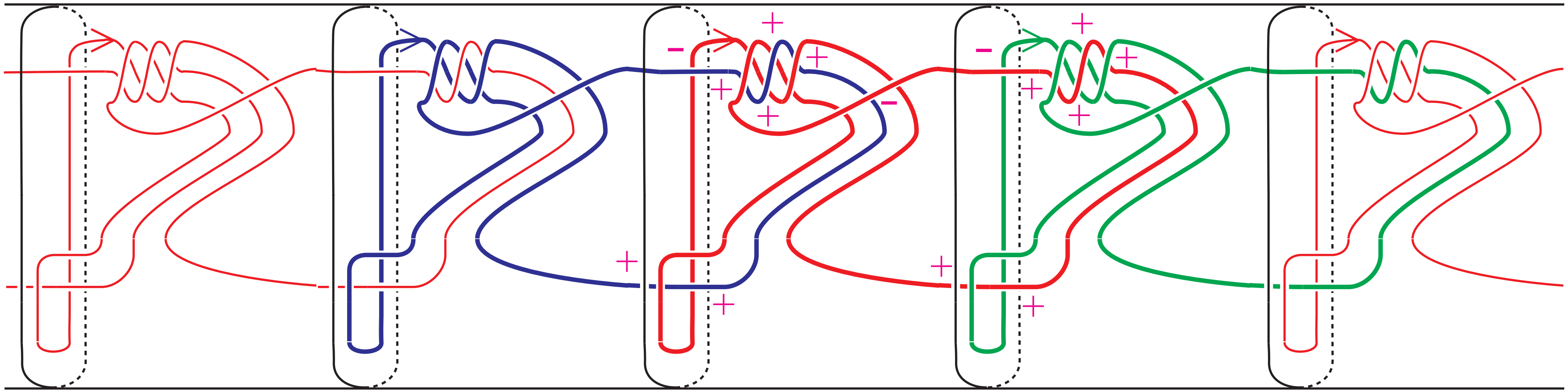


Definition. We define a polynomial $f(x)$ by $f(x) = \sum_{m \neq 0} lk(t^m \widetilde{L}, \widetilde{L}) x^m$.



$$\mathrm{lk}(t^{-1}\widetilde{L}, \widetilde{L}) = 2 \qquad \mathrm{lk}(t\widetilde{L}, \widetilde{L}) = 2$$

$$\mathrm{lk}(t^{m}\widetilde{L}, \widetilde{L}) = 0 \ (m \neq 1, -1)$$

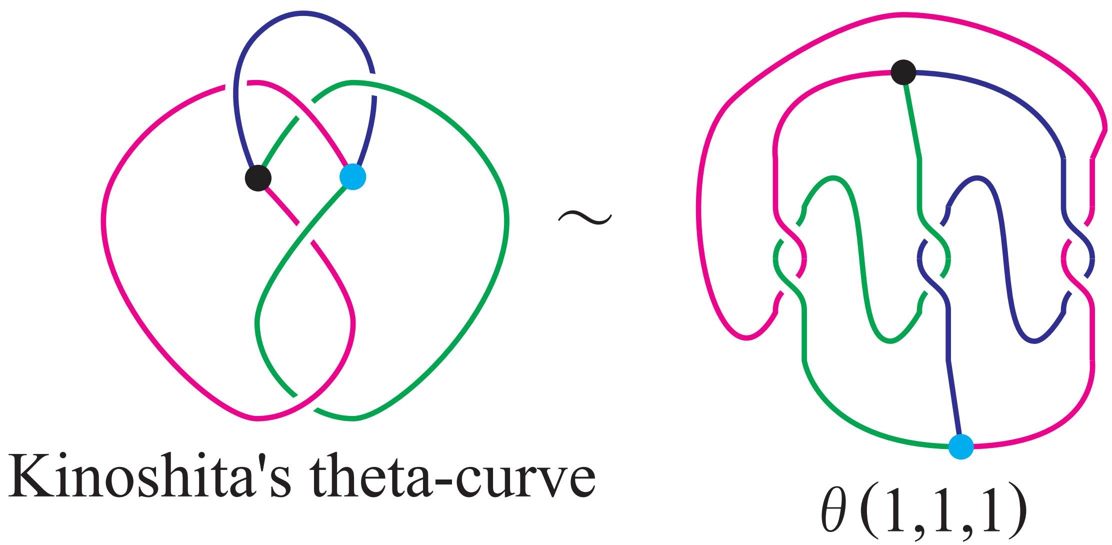
$$\Longrightarrow f(x) = 2x^{-1} + 2x$$

Remark. The polynomial $f(x)$ is related to the Kojima-Yamasaki η-function. For convenience, we reduce the constant term of the Kojima-Yamasaki invariant.

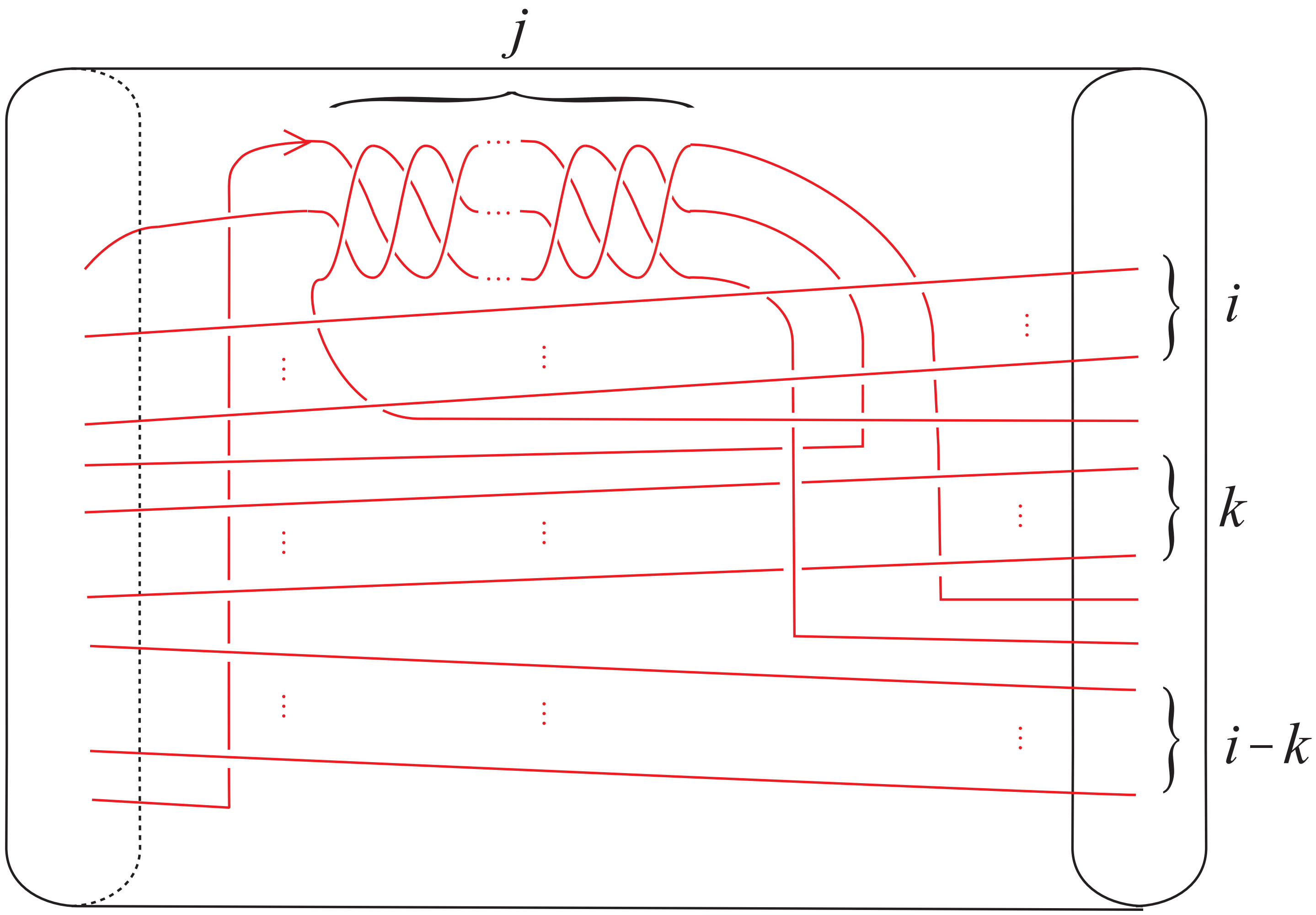
We have to consider three 2-component sublinks of the associated link, and obtain three polynomials $f_1(x)$, $f_2(x)$ and $f_3(x)$.

The multiset $\left\{f_1(x), f_2(x), f_3(x)\right\}$ is an ambient isotopy invariant of generalized Kinoshita's theta-curve.

Example 1. The multiset of Kinoshita's theta-curve $\theta(1, 1, 1)$ is $\left\{2x^{-1} + 2x, 2x^{-1} + 2x, 2x^{-1} + 2x\right\}$.



$(i > k > 1, j > 0)$



We must check some cases: $\underline{k > i > 1, j > 0}$, $\underline{k > i > 1, j > 0}$, and so on.

Theorem 1. If $\underline{i \neq j \neq k}$, the multiset of generalized Kinoshita's theta-curve $\theta(i,j,k)$ is

$$\Big\{ix^{-k} + ix^{-j} - ix^{j-k} - ix^{k-j} + ix^{j} + ix^{k},$$
$$jx^{-i} + jx^{-k} - jx^{k-i} - jx^{i-k} + jx^{k} + jx^{i},$$
$$kx^{-j} + kx^{-i} - kx^{i-j} - kx^{j-i} + kx^{i} + kx^{j}\Big\}.$$

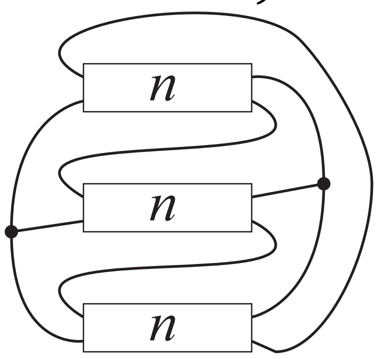
If $i = j$, the term $-kx^{i-j} - kx^{j-i}$ is vanished.

If $j = k$, the term $-ix^{j-k} - ix^{k-j}$ is vanished.

If $k = i$, the term $-jx^{k-i} - jx^{i-k}$ is vanished.

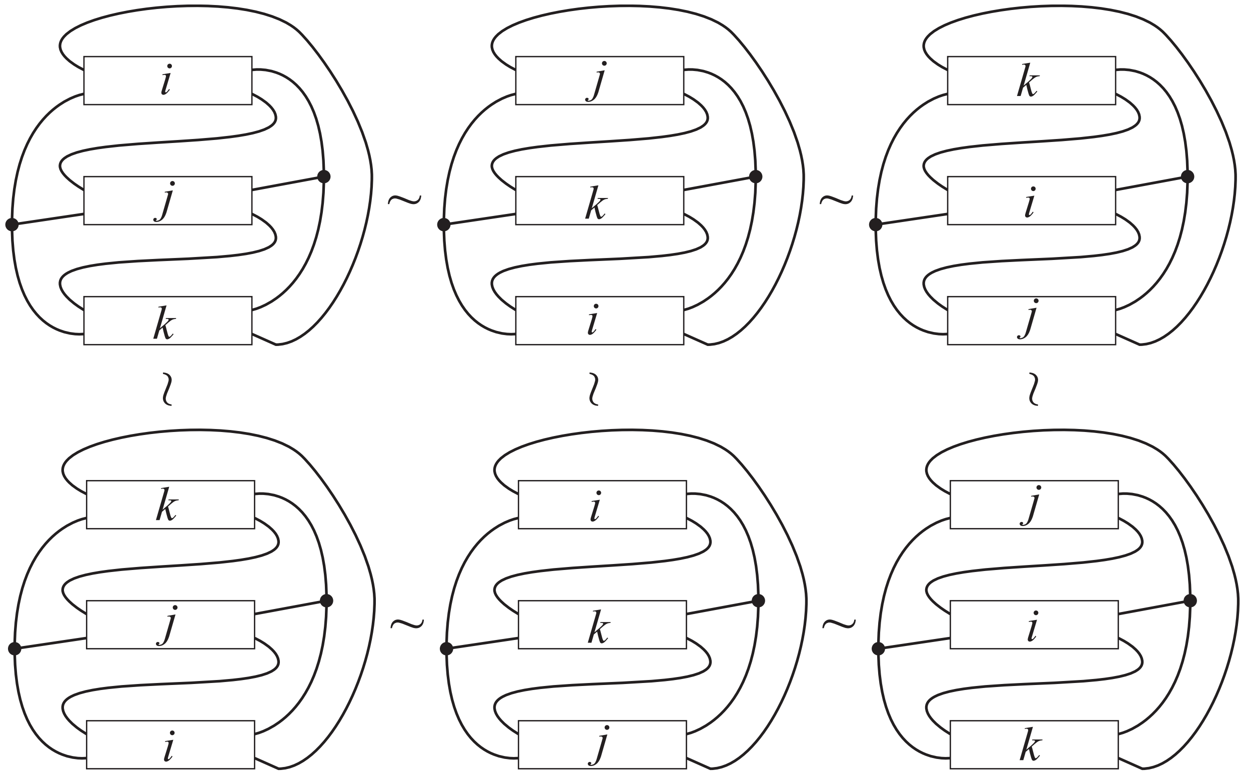
Moreover, the multiset of generalized Kinoshita's theta-curve $\theta(n,n,n)$ is

$$\Big\{2nx^{-1} + 2nx, 2nx^{-1} + 2nx, 2nx^{-1} + 2nx\Big\}.$$

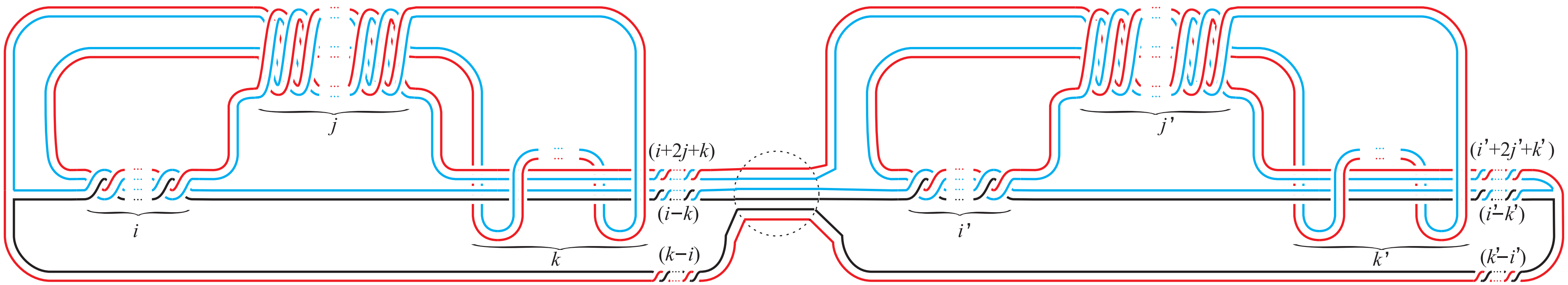


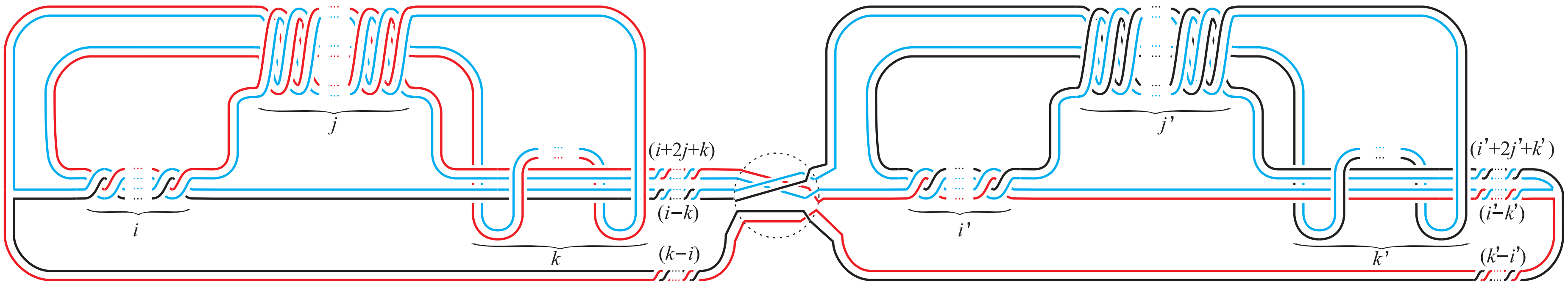
Corollary. Generalized Kinoshita's theta-curve $\theta(i, j, k)$ are mutually distinct up to the following symmetry:

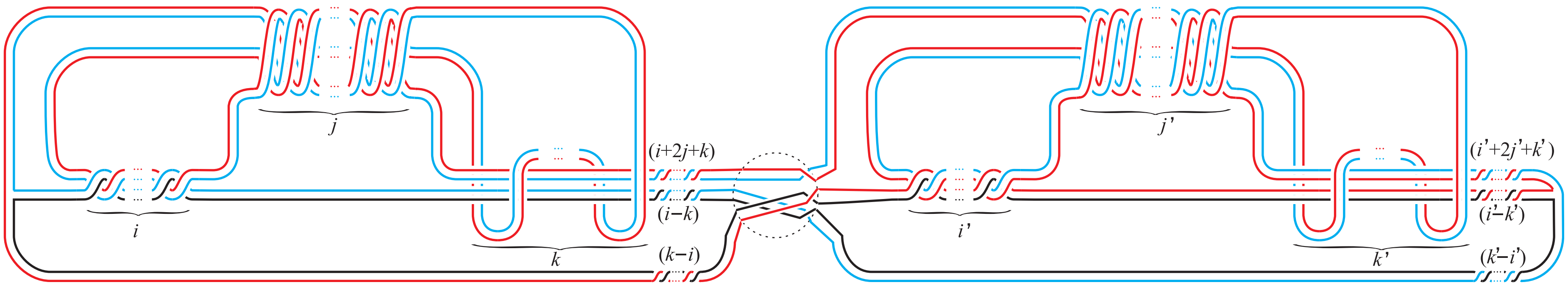
$$\theta(i, j, k) \sim \theta(j, k, i) \sim \theta(k, i, j) \sim \theta(j, i, k) \sim \theta(i, k, j) \sim \theta(k, j, i).$$

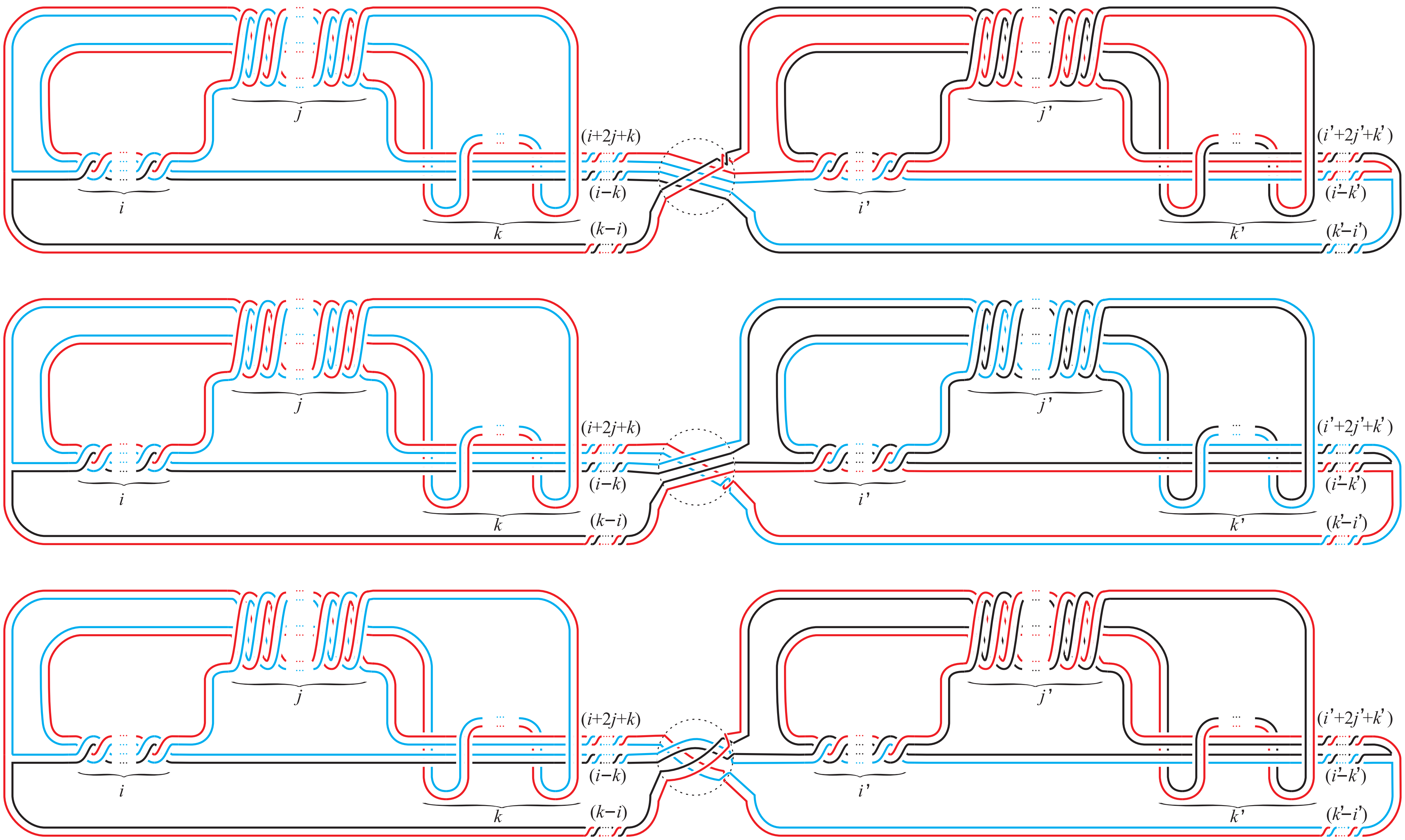


To calculate $f(x)$ of $\theta(i,j,k)\,\#_3\,\theta(i',j',k')$, we need to consider six different associated links.



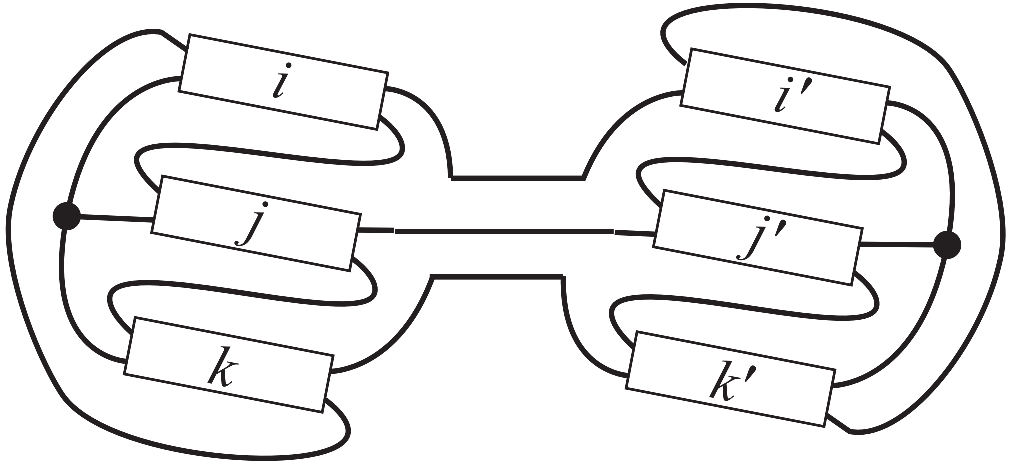






However, we can obtain the multiset of $\theta(i,j,k) \#_3 \theta(i',j',k')$ by adding each polynomials <u>in good order</u>.

$$\{ix^{-k} + ix^{-j} - ix^{j-k} - ix^{k-j} + ix^{j} + ix^{k},$$
$$jx^{-i} + jx^{-k} - jx^{k-i} - jx^{i-k} + jx^{k} + jx^{i},$$
$$kx^{-j} + kx^{-i} - kx^{i-j} - kx^{j-i} + kx^{i} + kx^{j}\}$$
$$+$$
$$\{i'x^{-k'} + i'x^{-j'} - i'x^{j'-k'} - i'x^{k'-j'} + i'x^{j'} + i'x^{k'},$$
$$j'x^{-i'} + j'x^{-k'} - j'x^{k'-i'} - j'x^{i'-k'} + j'x^{k'} + j'x^{i'},$$
$$k'x^{-j'} + k'x^{-i'} - k'x^{i'-j'} - k'x^{j'-i'} + k'x^{i'} + k'x^{j'}\}.$$



Theorem 2. If $\underline{i \neq j \neq k}$ and $\underline{i' \neq j' \neq k'}$, the multiset of $\theta(i,j,k) \#_3 \theta(i',j',k')$ is

$$\Big\{i(x^{-k}+x^{-j}-x^{j-k}-x^{k-j}+x^{j}+x^{k})+i'(x^{-k'}+x^{-j'}-x^{j'-k'}-x^{k'-j'}+x^{j'}+x^{k'}),$$
$$j(x^{-i}+x^{-k}-x^{k-i}-x^{i-k}+x^{k}+x^{i})+j'(x^{-i'}+x^{-k'}-x^{k'-i'}-x^{i'-k'}+x^{k'}+x^{i'}),$$
$$k(x^{-j}+x^{-i}-x^{i-j}-x^{j-i}+x^{i}+x^{j})+k'(x^{-j'}+x^{-i'}-x^{i'-j'}-x^{j'-i'}+x^{i'}+x^{j'})\Big\}.$$

Example 2. For $\theta(1,2,3)\,\#_3\,\theta(4,5,6)$, the multiset is

$$\Big\{(x^{-3}+x^{-2}-x^{-1}-x+x^2+x^3)+4(x^{-6}+x^{-5}-x^{-1}-x+x^5+x^6),$$
$$(2x^{-1}+2x^{-3}-2x^2-2x^{-2}+2x^3+2x)+5(x^{-4}+x^{-6}-x^2-x^{-2}+x^6+x^4),$$
$$(3x^{-2}+3x^{-1}-3x^{-1}-3x+3x+3x^2)+6(x^{-5}+x^{-4}-x^{-1}-x+x^4+x^5)\Big\}$$
$$=\Big\{4x^{-6}+4x^{-5}+x^{-3}+x^{-2}-5x^{-1}-5x+x^2+x^3+4x^5+4x^6,$$
$$5x^{-6}+5x^{-4}+2x^{-3}-7x^{-2}+2x^{-1}+2x-7x^2+2x^3+5x^4+5x^6,$$
$$6x^{-5}+6x^{-4}+3x^{-2}-6x^{-1}-6x+3x^2+6x^4+6x^5\Big\}.$$

For $\theta(1,2,3)\,\#_3\,\theta(5,4,6)$, the multiset is

$$\Big\{(x^{-3}+x^{-2}-x^{-1}-x+x^2+x^3)+5(x^{-6}+x^{-4}-x^{-2}-x^2+x^4+x^6),$$
$$(2x^{-1}+2x^{-3}-2x^2-2x^{-2}+2x^3+2x)+4(x^{-5}+x^{-6}-x-x^{-1}+x^6+x^5),$$
$$(3x^{-2}+3x^{-1}-3x^{-1}-3x+3x+3x^2)+6(x^{-4}+x^{-5}-x-x^{-1}+x^5+x^4)\Big\}$$
$$=\Big\{5x^{-6}+5x^{-4}+x^{-3}-4x^{-2}-x^{-1}-x-4x^2+x^3+5x^4+5x^6,$$
$$4x^{-6}+4x^{-5}+2x^{-3}-2x^{-2}-2x^{-1}-2x-2x^2+2x^3+4x^5+4x^6,$$
$$6x^{-5}+6x^{-4}+3x^{-2}-6x^{-1}-6x+3x^2+6x^4+6x^5\Big\}.$$

Since these sets are different, $\theta(1,2,3)\,\#_3\,\theta(4,5,6)\not\sim\theta(1,2,3)\,\#_3\,\theta(5,4,6)$.

For nonzero integers i, j, k, i', j' and k', we can classify almost

$\theta(i,j,k)\,\#_3\,\theta(i',j',k')$ by the multiset.

Remark. For $\theta(i,j,k) \#_3 \theta(-i,-j,-k)$, the multiset is $\Big\{0,0,0\Big\}$. Therefore, we can not detect the difference between $\theta(i,j,k) \#_3 \theta(-i,-j,-k)$ and $\theta(s,t,u) \#_3 \theta(-s,-t,-u)$ for all nonzero integers i, j, k, s, t and u by the multiset.

Example 3. We can not detect the difference between $\theta(1,1,1) \#_3 \theta(-1,-1,-1)$ and $\theta(2,2,2) \#_3 \theta(-2,-2,-2)$ by the multiset. However, their Yamada polynomials are different.

Hence, $\theta(1,1,1) \#_3 \theta(-1,-1,-1)$ is not ambient isotopic to $\theta(2,2,2) \#_3 \theta(-2,-2,-2)$.