The Annulus and Disk Complex is Contractible

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Background

• **Curve complex**:

Let S be an orientable closed connected surface with $g(S) > 1$.

Harvey defined curve complex $\mathcal{C}(S)$ of S as follows:

vertices : isotopy classes of ess. s.c.c.s on S

simplices : k+1 vertices determine a k-simplex

iff. they can be represented by disjoint curves.

Properties of $\mathcal{C}(\mathbf{S})$.

1.$\mathcal{C}(S)$ is connected and is a flag complex, (i.e.a set of vertices determines a simplex iff any two vertices are connected by an edge) .

2.(Harer) $\mathcal{C}(S)$ is homotopy equivalent to a bouquet of spheres $S^{(-\chi(S))}$.

3.(Masur and Minsky) $\mathcal{C}(S)$ is a (δ-)hyperbolic space.

• **Disk complex**:

Let S be the boundary of a handlebody H.

Hempel(?) defined the disk complex $\mathcal{D}(H)$ as follows:

vertices : isotopy classes of ess. s.c.c.s on S bounding disks in H

simplices : $k+1$ vertices determine a k-simplex

iff. they can be represented by disjoint curves.

Properties of $\mathcal{D}(\mathbf{H})$

1.$\mathcal{D}(S)$ is also connected and is a flag complex.

2.(McCullough)$\mathcal{D}(S)$ is contractible.

3.(Masur and Schleimer)$\mathcal{D}(S)$ is also a (δ-)hyperbolic space.

Definition of annulus and disk complex

Let C be a compression body and S be the plus boundary of C. (For convention, we say that an ess. s.c.c c on S bounds an spanning annulus A if it is a boundary component of A).

Then we define the annulus and disk complex $\mathcal{AD}(C)$ as follows:

$\mathbf{vertices}$: isotopy classes of ess. s.c.c on S bounding essential disks or spanning annuli.

$\mathbf{simplices}$: $k{+}1$ vertices determine a k-simplex

iff. they can be represented by disjoint curves.

Properties of $\mathcal{AD}(\mathbf{C})$.

1.$\mathcal{AD}(C)$ is a subcomplex of $\mathcal{C}(S)$.

2.If C is a trivial compression body, then $\mathcal{AD}(C)$ is just $\mathcal{C}(S)$.
If C is a handlebody, then $\mathcal{AD}(C)$ becomes of $\mathcal{D}(C)$

3.$\mathcal{AD}(C)$ is also a connected flag complex.

• **Natural question**: What's the topological and geometric property of $\mathcal{AD}(C)$? (homotopy type? δ-hyperbolic space?)

Using McCullough's method, we prove that

- Theorem:

$\mathcal{AD}(C)$ is contractible if C is non-trivial.

Before the proof, we define another simplex $\widetilde{\mathcal{AD}(C)}$ which is isomorphic to $\mathcal{AD}(C)$:

vertices : isotopy classes of essential disks and spanning annuli in C

simplices : $k+1$ vertices determine a k-simplex

iff. they can be represented by disjoint surfaces.

Not hard to see $\widetilde{\mathcal{AD}(C)}$ is isomorphic to $\mathcal{AD}(C)$. So only necessary to prove $\widetilde{\mathcal{AD}(C)}$ is contractible.

Sketch of Proof

- Some basic lemmas

1.A CW-compelx X is contractible, if and only if any map from sphere S^q to X is null-homotopic for any integer $q \geq 0$.

2.Suppose A is a subspace of X, and A is contractible. Then any map $f : Y \to X$ satisfying $f(Y) \subset A$ is null-homotopic.

3. Suppose a is a vertex of a simplical complex X.
Let $St_a X$ be the star neighborhood of a in X and denote the closure of $St_a X$ in X by $\overline{St_a X}$. Then $\overline{St_a X}$ is contractible.

Let $X = \widetilde{\mathcal{AD}(C)}$.

Idea :

For any map $f : S^q \to X$,

find another map g,

s.t. g is homotopic to f and $g(S^q) \subset \overline{St_a X}$ for some $a \in X^{(0)}$.

To realize this idea, first choose an essential disk E of C, and let $a = [E]$ be a vertex of X (This is why we require that C is non-trivial).

Define a space $\Omega := \{(K, g) : K$ is a triangulation of S^q, and $g : (S^q, K) \to X$ is simplical and is homotopic to $f\}$

Define a complexity function P on Ω as follows:
$\forall (K, g) \in \Omega$,
$P_i(K, g) := \sharp$ of $\{v \in K^0 : g(v) \cdot a = i\}$..
$P(K, g) := (..., P_3(K, g), P_2(K, g), P_1(K, g))$,
and the complexities are ordered lexicographically.

Claim: If (K, g) realize the minimal complexity of P in Ω, then
$P(K, g) = (...0, ..., 0)$.

Claim $\Rightarrow$
each vertex v of K is mapped into $\overline{St_a X}$ (since $g(v) \cdot a = 0$)$\Rightarrow$
$g(K) \subset \overline{St_a X}$ (since g is a simplical map).

Proof of Claim

Suppose $P_i(K,g)=0$ for all $i>n$ and $P_n(K,g)>0$ for some $n>0$.

Choose a vertex $v_0 \in K$ such that $g(v_0)\cdot a = n$.

Let $v_1, v_2, ..., v_k$ be the vertices in K adjacent to v_0.

Then

$\exists$ representatives F_i for $g(v_i)$ each i

s.t.

1.F_i intersects F_j minimally for $i \neq j$, i.e. $|F_i \bigcap F_j| = [F_i]\cdot[F_j]$;

2.Each F_i intersects E minimally, i.e. $|F_i \bigcap E| = [F_i]\cdot[E]$.

$n > 0 \Rightarrow F_0 \bigcap E \neq \emptyset$.
Consider an outermost arc α of $F_0 \bigcap E$ on E.
There is an outermost disc $B \subset E$ such that $\partial B \subset \alpha \bigcup \partial E$ and $int(B) \bigcap F_0 = \emptyset$.

Then there are two possibilities.

Case 1. $B \bigcap (\bigcup_{i=1}^{k} F_i) = \emptyset$.

Case 2. $B \bigcap (\bigcup_{i=1}^{k} F_i) \neq \emptyset$.

Proof of Case 1

Surgery F_0 along B, get another ess disk or spanning annulus $F_0^{'}$. Then $[F_0^{'}] \cdot [E] < [F_0] \cdot [E]$.
Let $\widehat{g}$ be a map defined on vertices of K by $\widehat{g}(w) = g(w)$ if $w \neq v_0$ and $\widehat{g}(v_0) = [F_0^{'}]$.

Then we can extend $\widehat{g}$ to an simplicial map from (S^q, K) to X by linear extension. (This can be done since all vertices of a simplex of K are carried into a common simplex of X by $\widehat{g}$).

We can find a homotopy g_t such that
$g_0 = g, g_1 = \widehat{g}, g_t|_{K^0-\{v_0\}} = g|_{K^0-\{v_0\}}$
and $g_t(v_0)$ moves from $[F_0]$ to $[F_0^{'}]$ on the 1-simplex $\langle[F_0][F_0^{'}]\rangle$ when t increases from 0 to 1.
Each g_t can be extend to whole K by linear extension, since all vertices of a simplex of K are carried into a common simplex of X by each g_t.

Not hard to see $P(K, \widehat{g}) < P(K, g)$. Contradiction!

The proof of the second case is similar (through more complicated).

End

THANK YOU !