Whitney tower concordance and Casson-Gordon style invariants of links

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The 8th EAST Asian school of knots and related topics

Definition

A (relative) cobordism $(W^4; M, M')$ $(\partial W = M \cup_\partial -M')$ is called *homology cobordism* if $H_i(M) \cong H_i(W) \cong H_i(M')$ for all i,

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Example

If two links L and L' are concordant via concordance C, then

$$(S^3 \times [0,1] - \nu(C); S^3 - \nu(L), S^3 - \nu(L'))$$

is homology cobordism.

Remark

If W is a *homology cobordism* between exteriors of L and L', with $\pi_1(W)$ normally generated by meridians, then L and L' are (topologically) concordant.

Whitney concordance and solvable cobordism

Knots and Links

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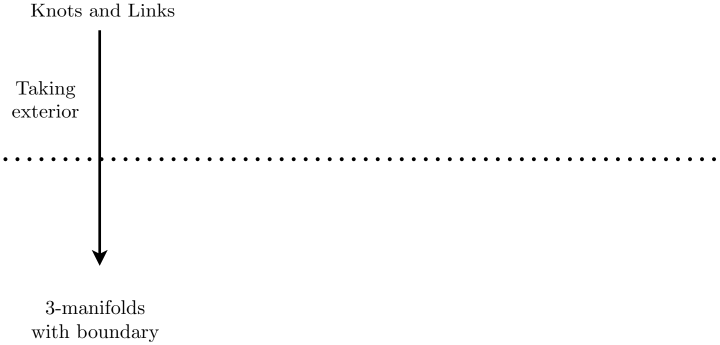
Whitney concordance and solvable cobordism



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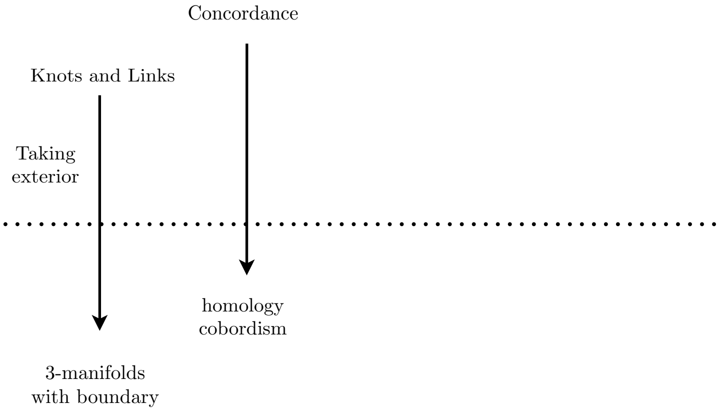

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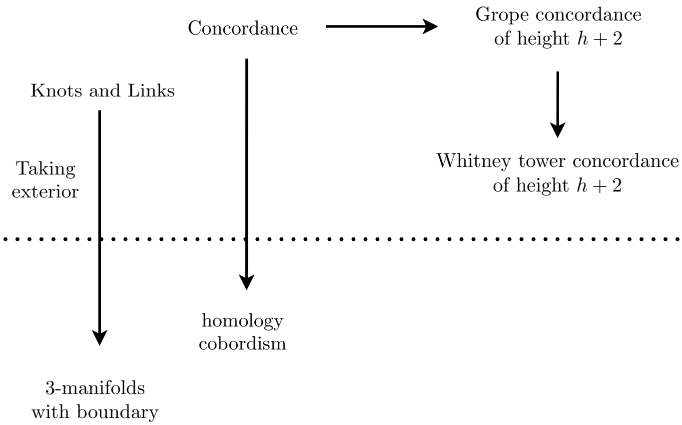

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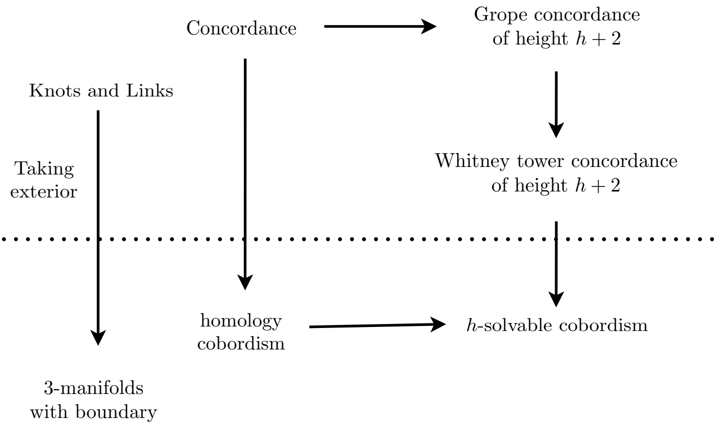

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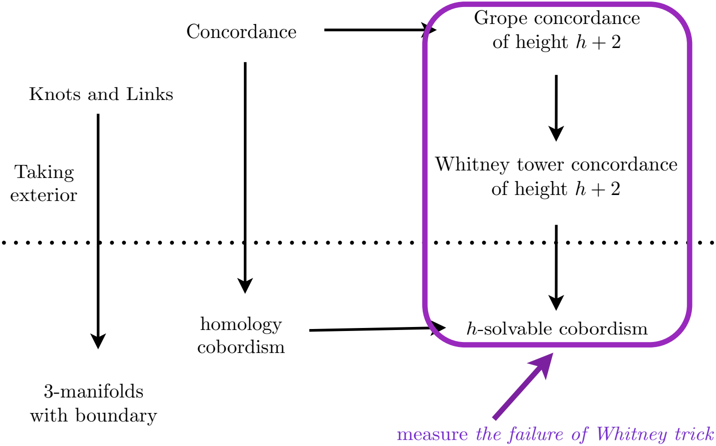

Let $(W; M, M')$ be a H_1-cobordism and $\pi = \pi_1(W)$ with derived series

$$\pi^{(0)} = \pi, \quad \pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}].$$

Following C. T. C. Wall,

$$\lambda_n \colon H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \longrightarrow \mathbb{Z}[\pi/\pi^{(n)}],$$

$$\mu_n \colon H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \longrightarrow \mathbb{Z}[\pi/\pi^{(n)}]/\langle g - \bar{g} \rangle$$

Definition of n-Lagrangians and k-duals for $k \leq n$

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2. For $n \geq k$ and n-Lagrangian $l_1, \ldots, l_r$,
$$d_1, \ldots, d_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(k)}]) \text{ is a } k\text{-dual}$$
if $\lambda_k(l_i, d_j) = \delta_{ij}$.

Definition of *h*-solvable cobordism, $h \in \frac{1}{2}\mathbb{N} \cup \{0\}$.

Definition (Cha)

Under the above condition, $(W; M, M')$ is called an

- *n*-solvable cobordism if it has *n*-Lagrangian with *n*-dual.
- *n*.5-solvable cobordism if it has $(n+1)$-Lagrangian with *n*-dual.

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- From our convention, 1-Lagrangian, 1-dual *live* in second homology of abelian cover.

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- 2-Lagrangian *lives* in second homology of abelian cover of abelian cover.

Convention

Now L denotes 2-component link with linking number 1.

$$X_L = S^3 - \nu(L), \Lambda = \mathbb{Z}[\mathbb{Z}^2].$$

Classical abelian invariants

Theorem (Kawauchi)

If L is concordant to the Hopf link H, then

$$\Delta_L(s,t) = f(s,t)f(s^{-1},t^{-1}), \quad f(s,t) \in \Lambda$$

up to multiplication of the unit of Λ.

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Theorem (K)

If L is height 3 Whitney tower concordant to the Hopf link H, then

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Theorem (Hillman)

If L is concordant to the Hopf link H, then the Blanchfield form

$$b_L \colon tH_1(X_L;\Lambda) \times tH_1(X_L;\Lambda) \longrightarrow K/\Lambda$$

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Theorem (Friedl-Powell '11)

If L is concordant to H, then $\tau(L, \varphi, \chi) = 0 \in L^0(\mathcal{K}) \otimes \mathbb{Q}$.

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1. $\tau(L, \varphi, \chi)$ is a 2-component link (with linking number 1) version of *remarkable* work of Casson-Gordon invariant [Casson-Gordon ′86].

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Nontriviality of the result

Remark (Cha)

For any integer $n > 2$, there are links with two unknotted components of linking number 1, which are height n Whitney tower concordant to H, but not height $n.5$ Whitney tower concordant to H.

Goal

- *Give a sketch of the proof for classical abelian invariants.*
- *Discuss the construction of Friedl-Powell invariants and ideal of the proof.*

Classical argument

$W = S^3 \times [0,1] - \nu(C)$, C: concordance between L and H.

$$tH_2(W, \partial W; \Lambda) \xrightarrow{\partial} tH_1(\partial W; \Lambda) \xrightarrow{\text{inc}_*} tH_1(W; \Lambda)$$

is exact from the fact:

- $H_2(W, \partial W; \Lambda) = tH_2(W, \partial W; \Lambda)$

(Here, tM denotes the Λ-torsion submodule of M.)
Then,

$$P = \text{Ker}(tH_1(\partial W; \Lambda) \xrightarrow{\text{inc}_*} tH_1(W; \Lambda))$$

satisfies $P = P^{\perp}$ for b_L.

Lemma (K)

$(W^4; M, M')$: cobordism, a ring homomorphism $\mathbb{Z}[\pi_1 W] \to R$. If $(W; M, M')$ is R-coefficient H_1-cobordism and R has the quotient field K, then $\beta_2^R(W, M) = \chi^R(W, M)$.

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Proof of the Lemma.

From the UCSS with $C_* = C_*(W, M'; R)$,

$$E_2^{pq} = \mathrm{Ext}_R^q(H_p(C_*), K) \Rightarrow H^*(C_*; K)$$

$$\mathrm{Ext}_R^q(M, K) = \mathrm{Ext}_K^q(M \otimes_R K, K) = 0 \textit{ if } q \geq 1.$$

Hence, $H^n(W, M'; K) = \mathrm{Hom}_K(H_n(W, M'; R) \otimes_R K, K)$.
For $i = 3, 4$, we have

$$H_i(W, M; K) = H^{4-i}(W, M'; K) = \mathrm{Hom}_K(H_{4-i}(W, M'; R) \underset{R}{\otimes} K, K) = 0$$

□

Proof of the Theorem.

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Applying Lemma twice, we have

$$\beta_2^K(W, X_L) = \chi^K(W, X_L) = \chi(W, X_L) = \beta_2(W, X_L).$$

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Let $l_1, \ldots, l_r, d_1, \ldots, d_r$ be images of 1-Lagrangians, 1-duals via

$$j\colon H_2(W; \Lambda) \to H_2(W, X_L; \Lambda) = H_2(W, \partial W; \Lambda).$$

Note that $2r = \beta_2(W, X_L) = \beta_2^K(W, X_L)$ and $j \otimes_\Lambda K$ is surjective.

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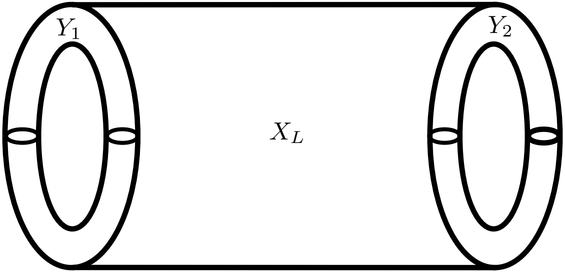
Diagram chasing shows

$$tH_2(W, \partial W; \Lambda) \xrightarrow{\partial} tH_1(\partial W; \Lambda) \xrightarrow{\text{inc}_*} tH_1(W; \Lambda)$$

is exact and this completes the proof. □

Construction of Friedl-Powell invariant



$$Y = Y_1 \cup Y_2, \quad M_L = X_L \cup \partial X_H \times [0,1] \cup X_H.$$

Construction of Friedl-Powell invariant

Choose $\varphi \colon H_1(M_L) \to \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ for a prime p.

$$\begin{array}{ccc} H_1(M_L) & \xrightarrow{\ \varphi\ } & \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j \\ {\scriptstyle \text{res}}\downarrow & \nearrow{\scriptstyle \varphi|} & \\ H_1(X_L) = \mathbb{Z} \oplus \mathbb{Z} & & \end{array}$$

with $\varphi| \colon (1,0) \mapsto (1,0)$
$(0,1) \mapsto (0,1)$

$M_L^{\varphi}, X_L^{\varphi}, Y^{\varphi}$ are $\mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$-cover of M_L, X_L, Y from φ.

Construction of Friedl-Powell invariant

Choose another prime q and homomorphisms

$$\psi \colon H_1(M_L^{\varphi}) \longrightarrow H_1(M_L) = \mathbb{Z}^3, \quad \chi \colon H_1(M_L^{\varphi}) \longrightarrow \mathbb{Z}/q^l$$

$$\begin{array}{ccc} M_L^{\varphi} \amalg M_H^{\varphi} & \longrightarrow & \mathbb{Z}^3 \times \mathbb{Z}/q^l \\ \big\downarrow & \nearrow & \\ {}^{\exists} W & & \end{array}$$

$$\partial W = s(M_L^{\varphi} \amalg -M_H^{\varphi}) for some s \in \mathbb{N}$$

There is a ring homomorphism

$$\alpha \colon \mathbb{Z}[\mathbb{Z}^3 \times \mathbb{Z}/q^l] \longrightarrow \mathbb{Q}(\xi_{q^l})[\mathbb{Z}^3] \longrightarrow \mathcal{C}(\mathbb{Z}^3) = \mathcal{K}.$$

Definition (Friedl-Powell)

$$\tau(L, \varphi, \chi) = \frac{1}{s}([W_{\mathcal{K}}] - [W_{\mathbb{Q}}]) \in L^0(\mathcal{K}) \otimes \mathbb{Q}$$