Invariants for handlebody-knots derived from Yetter–Drinfeld modules

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Def

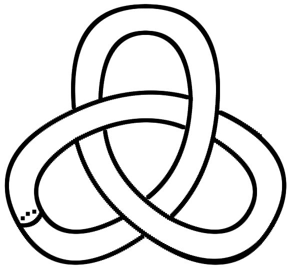
- A (genus g) handlebody-knot is

  a handlebody (of genus g) $\hookrightarrow S^3$

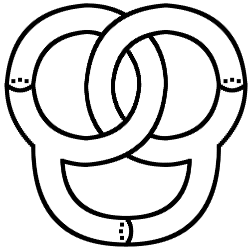
- A handlebody-link is

  a disjoint union of handlebodies $\hookrightarrow S^3$

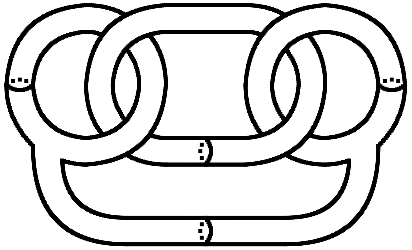
Example

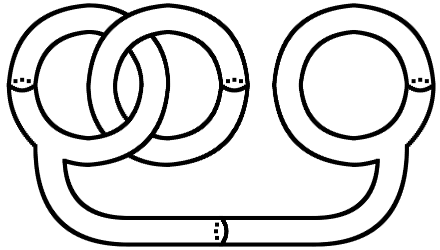


a genus $\mathbf{1}$ handlebody-knot



a genus $\mathbf{2}$ handlebody-knot





handlebody-links

Def Handlebody-links $H_1 = H_2$

$$\overset{\text{def}}{\Leftrightarrow} \exists f : S^3 \to S^3 : \text{ori. pre. homeo. s.t. } f(H_1) = H_2$$

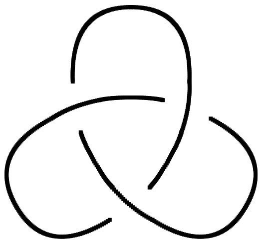
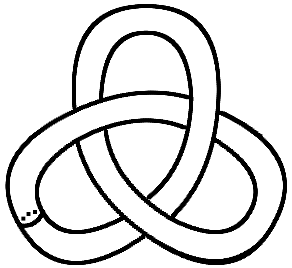
$$\Leftrightarrow \exists f_t : S^3 \to S^3 : \text{iso. s.t. } f_0 = \mathbf{id},\ f_1(H_1) = H_2$$

Handlebody-knot theory is a generalization of knot theory

$\{$ knot $\} \to \{$ handlebody-knot $\}$ is a well-defined injection.

$$\boldsymbol{K} \mapsto \boldsymbol{N}(\boldsymbol{K}) \quad \text{(a reg. n.b.d. of } \boldsymbol{K}\text{)}$$

(with $\boldsymbol{K} \in \{$ knot $\}$ and $\boldsymbol{N}(\boldsymbol{K}) \in \{$ handlebody-knot $\}$)

 $\mapsto$ 

A handlebody-link is the n.b.d. equiv. class of a spatial graph

Def [Suzuki] Spatial graphs $\boldsymbol{K_1} \overset{\text{nbd}}{\sim} \boldsymbol{K_2}$

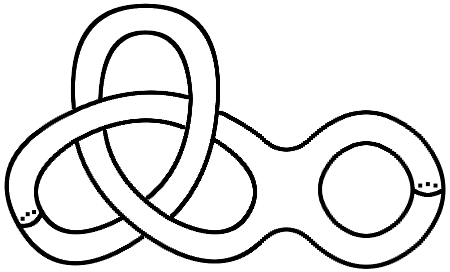
$\overset{\text{def}}{\Leftrightarrow} \exists \boldsymbol{f} : \boldsymbol{S^3} \to \boldsymbol{S^3}$: ori. pre. homeo. s.t. $\boldsymbol{f}(\boldsymbol{N}(\boldsymbol{K_1})) = \boldsymbol{N}(\boldsymbol{K_2})$

<u>Def</u> $\boldsymbol{H}$: a handlebody-link

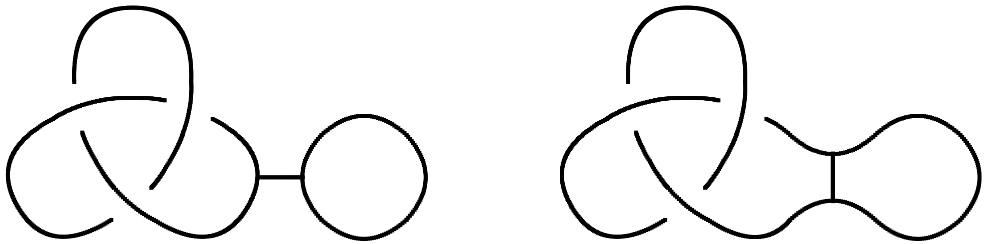
$\boldsymbol{K}$: a spatial graph satisfying $\boldsymbol{H} = \boldsymbol{N}(\boldsymbol{K})$

$\boldsymbol{D}$: a diagram of $\boldsymbol{K}$

Then, we say that
- $\boldsymbol{H}$ is represented by $\boldsymbol{K}$, and
- $\boldsymbol{D}$ is a diagram of $\boldsymbol{H}$.



a handlebody-link



diagrams

<u>Remark</u>

$$
\begin{array}{ccccc}
\{\text{ sp. gr. }\} & \hookleftarrow & \{\text{ sp. tri. gr. }\} & \hookleftarrow & \{\text{ sp. bridgeless tri. gr. }\} \\
\quad \boldsymbol{N} \searrow & & \downarrow & \swarrow & \uparrow \\
& & \{\text{ handlebody-link }\} & \twoheadleftarrow & \{\bigcirc, \Phi, \Phi\!\Phi, \ldots \hookrightarrow \boldsymbol{S}^3\}
\end{array}
$$

Thm [I] H_1, H_2 : handlebody-links

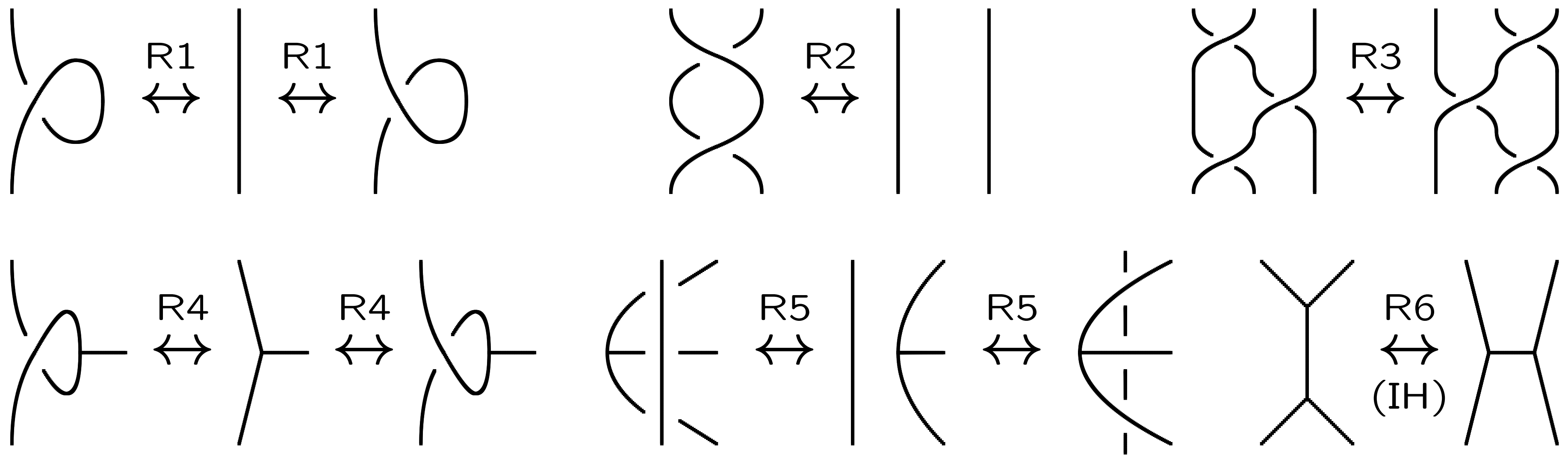
K_1, K_2 : spatial trivalent graphs representing H_1, H_2

D_1, D_2 : diagrams of H_1, H_2 (and K_1, K_2)

Then $H_1 = H_2 \Leftrightarrow K_1, K_2$ are related by IH-moves

$\Leftrightarrow D_1, D_2$ are related by R1–R6 moves

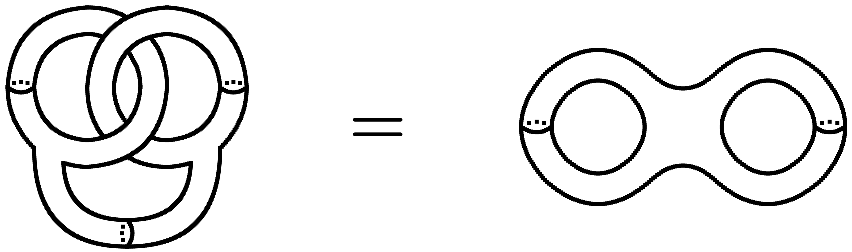
That is, {handlebody-link} = {spatial tri. graph} / IH-move.

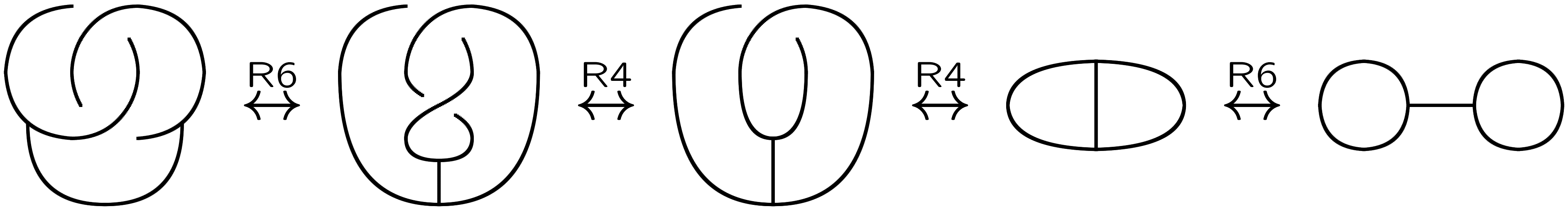


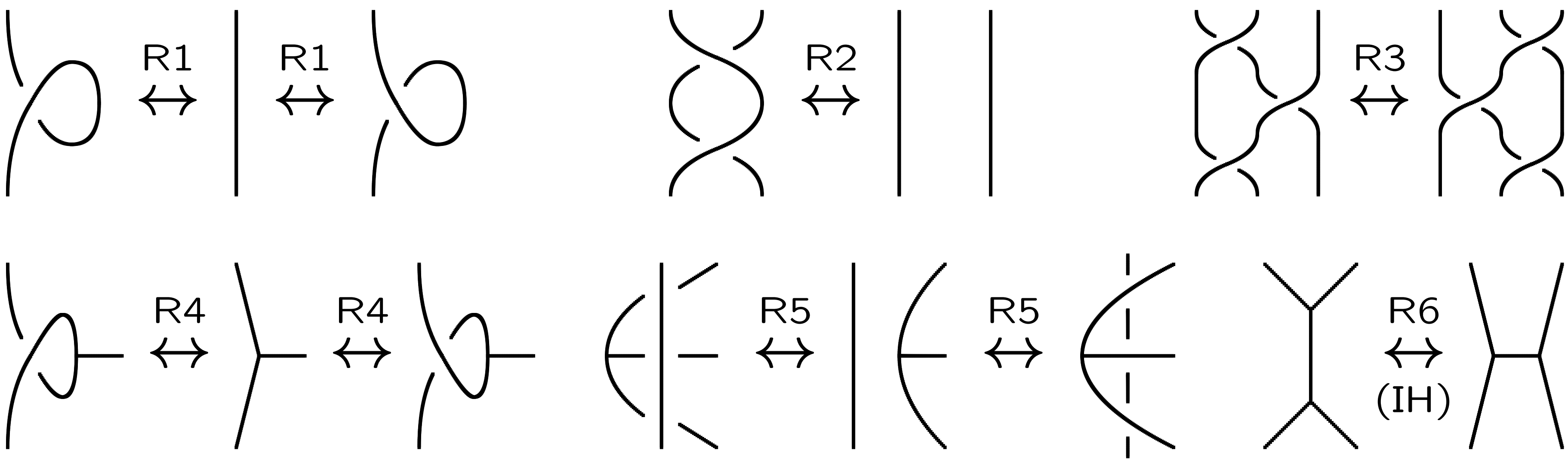
Thm [Ishihara–I]

{handlebody-link} = {spatial bridgeless tri. graph} / IH-move

Example







A table of irreducible genus 2 handlebody-knots up to 6 crossings [IKMS]

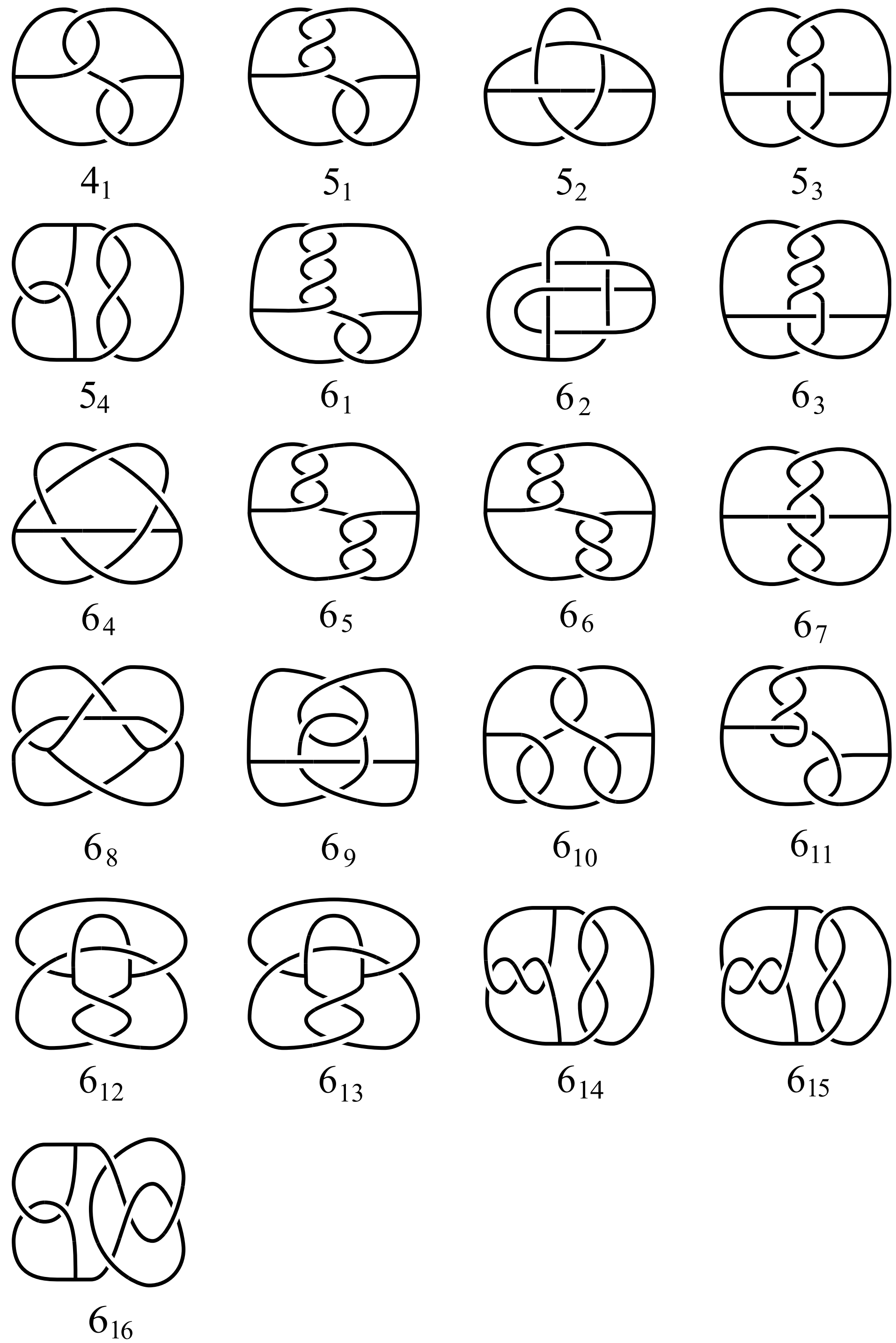
We remark that

- $\mathbf{5_1 \neq 6_4}$, $\mathbf{5_2 \neq 6_{13}}$

  [J. H. Lee–S. Lee]

- $\mathbf{6_{14} \neq 6_{15}}$

  [Kishimoto–Ozawa–I]



Invariants for handlebody-links

- an invariant of the exterior of a handlebody-link
- a quandle $\begin{cases}\text{coloring} \\ \text{cocycle}\end{cases}$ invariant with
  a G-family of quandles [Iwakiri–Jang–Oshiro–I]
- a quantum invariant derived from
  a unimoduler Hopf algebra [Masuoka–I]
- a quantum invariant derived from
  Yokota polynomial [Mizusawa–Murakami]

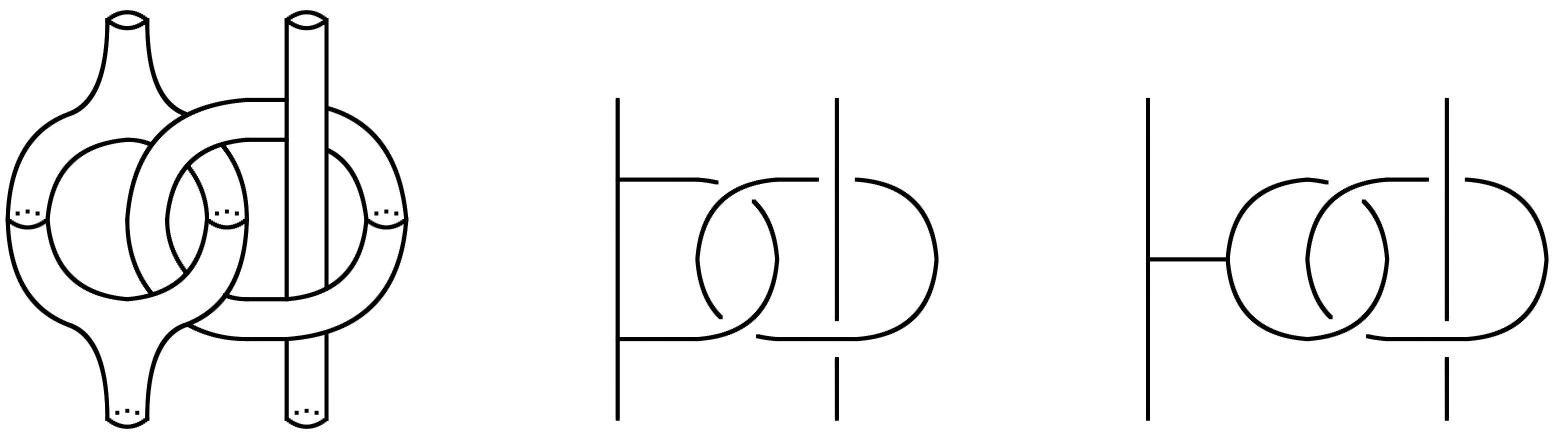
Def

- A handlebody-tangle is

  a disjoint union of handlebodies $\hookrightarrow \boldsymbol{B^3}$ whose intersection with $\boldsymbol{\partial B_3}$ consists of disks (*end disks*)

- A trivalent tangle is

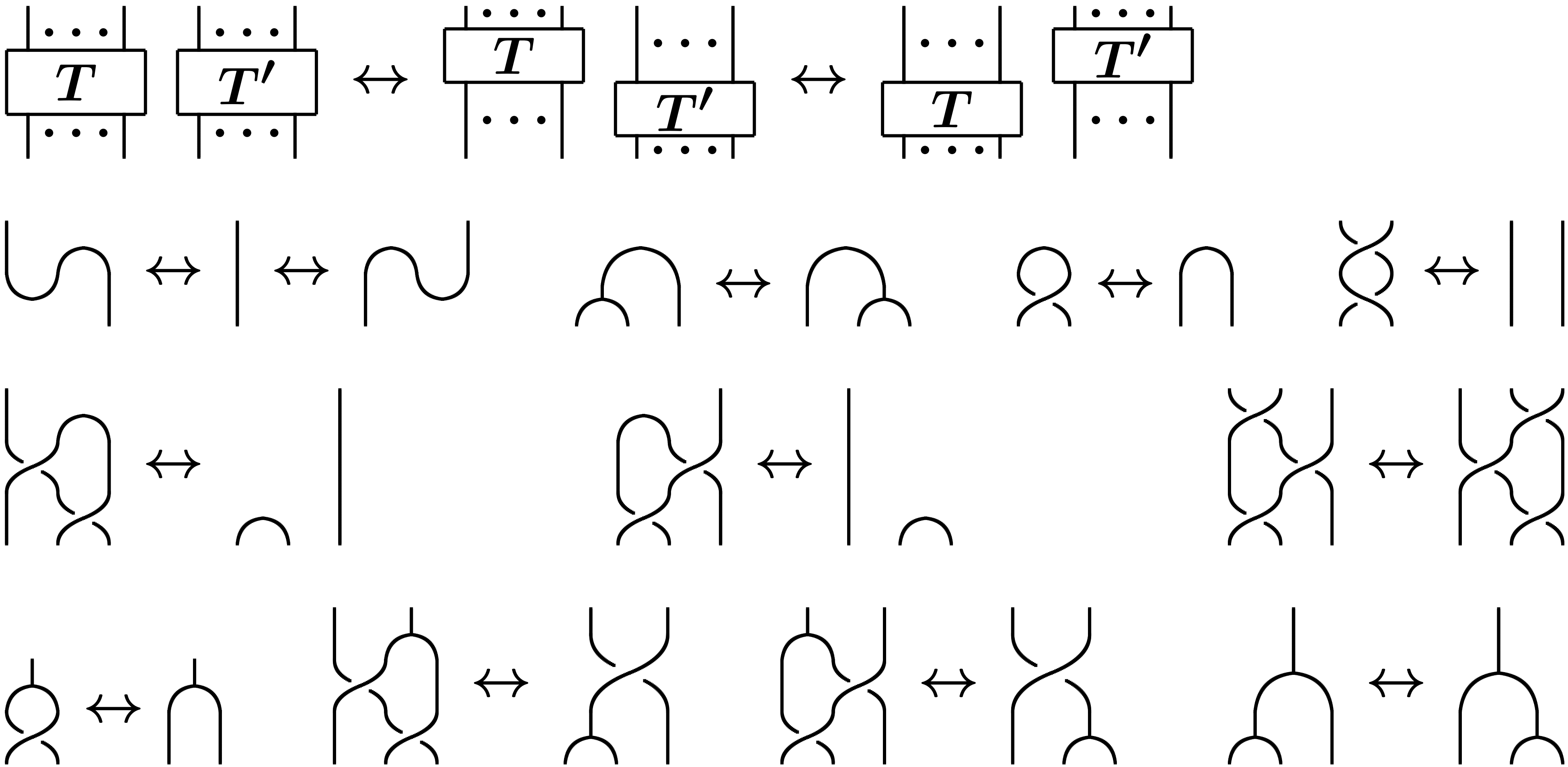
  a uni-trivalent graph $\hookrightarrow \boldsymbol{B^3}$ whose intersection with $\boldsymbol{\partial B_3}$ consists of all univalent vertices (*end points*)



Prop [Ishihara–I, Masuoka–I]

T_1, T_2 : handlebody-tangles represented by D_1, D_2

($D_1 = D_2$ if $D_1 \sim_{\mathrm{iso}} D_2$ preserving crossings, vertices, maxima, and minima.) Then $T_1 = T_2 \Leftrightarrow D_1, D_2$ are related by



$\mathcal{T}$: the category of handlebody-tangles

objects: finite sequences of a dot $\bullet$

ex. $\emptyset$, $\bullet$, $\bullet\ \bullet$, $\bullet\ \bullet\ \bullet$, . . .

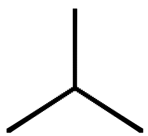
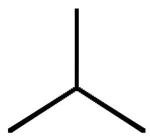
morphisms: isotopy classes of handlebody-tangles

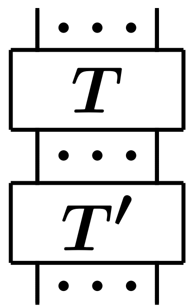
(represented by trivalent tangles)

identity: $\mathbf{id}_{\emptyset} = \emptyset, \quad \mathbf{id}_{\bullet} = \Big| \; , \quad \mathbf{id}_{\bullet\bullet} = \Big| \; \Big| \; , \ldots$

source: $s(T)$ of a morphism T is the bottom ends of T

target: $b(T)$ of a morphism T is the top ends of T

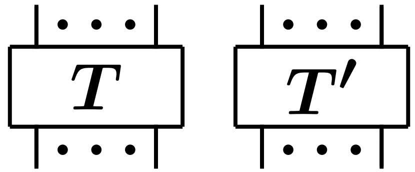
ex. s () $= \bullet\ \bullet$ $\qquad b$ () $= \bullet$

composition: $T \circ T' =$ 

$\mathcal{T}$ is a strict tensor category with

tensor product: $\underbrace{\bullet\cdots\bullet}_{m} \otimes \underbrace{\bullet\cdots\bullet}_{n} := \underbrace{\bullet\cdots\bullet}_{m+n}$

$$\boldsymbol{T} \otimes \boldsymbol{T'} :=$$



unit: $\emptyset$

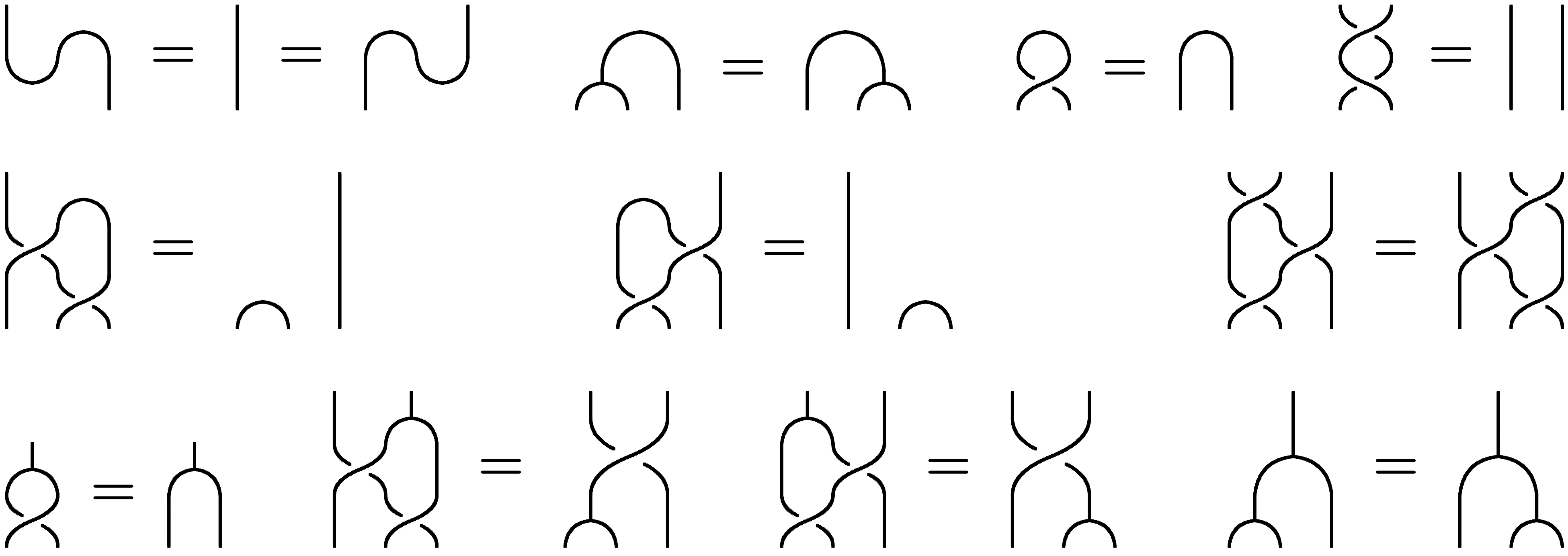
associativity constraint: identity

unit constraint: identity

Prop [Ishihara–I, Masuoka–I]

$\mathcal{T}$ is generated by the morphisms

and the relations

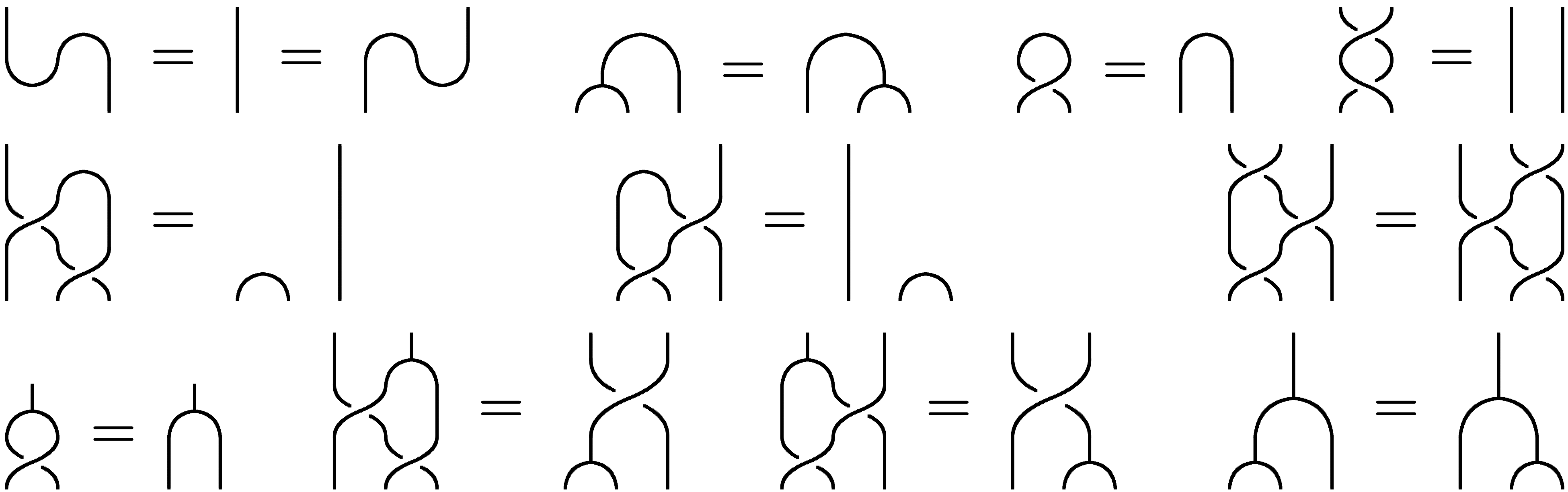


A functor $\boldsymbol{F} : \mathcal{T} \to \boldsymbol{\mathcal{V}ec_k}$ gives an inv. for handlebody-links.

$$\boldsymbol{F(H) : k \to k; 1 \mapsto v(H)}$$

$\boldsymbol{F} : \mathcal{T} \to \boldsymbol{\mathcal{V}ec_k}$ is a functor if

- $(\boldsymbol{F}(\,|\,) \otimes \boldsymbol{F}(\cap)) \circ (\boldsymbol{F}(\cup) \otimes \boldsymbol{F}(\,|\,)) = \boldsymbol{F}(\,|\,)$
- $(\boldsymbol{F}(\cap) \otimes \boldsymbol{F}(\,|\,)) \circ (\boldsymbol{F}(\,|\,) \otimes \boldsymbol{F}(\cup)) = \boldsymbol{F}(\,|\,)$
- $\cdots$



Thm [Masuoka–I]

A unimoduler finite-dimensional Hopf algebra yields an invariant for handlebody-links

$A = (A, \mu, \eta, \Delta, \varepsilon, S)$: a Hopf algebra over k

multiplication $\mu : A \otimes A \to A; a \otimes b \mapsto ab$
unit $\eta : k \to A; 1_k \mapsto 1_A$
comultiplication $\Delta : A \to A \otimes A; a \mapsto a_1 \otimes a_2$ $(= \sum_i a_1^i \otimes a_2^i)$
counit $\varepsilon : A \to k$ (Sweedler notation)
antipode $S : A \to A$

Def A : unimodular $\overset{\text{def}}{\Leftrightarrow} I_l(A) = I_r(A)$

$I_l(A) := \{\Lambda \in A \,|\, a\Lambda = \varepsilon(a)\Lambda \ (\forall a \in A)\}$ (left integrals)

$I_r(A) := \{\Lambda \in A \,|\, \Lambda a = \varepsilon(a)\Lambda \ (\forall a \in A)\}$ (right integrals)

A unimoduler finite-dimensional Hopf algebra A

F ( ) $:= \mathrm{id}_A : A \to A; a \mapsto a$

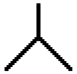
F () $: A \otimes A \to \mathbb{C}; a \otimes b \mapsto \lambda(ab)$ where $\lambda \in I_l(A^*)$

F () $: \mathbb{C} \to A \otimes A; 1 \mapsto \sum_i \beta_i \otimes \alpha_i$

where $\{\alpha_i\}$, $\{\beta_i\}$ are bases of A s.t. $\lambda(\alpha_i \beta_i) = \delta_{ij}$

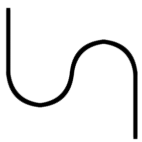
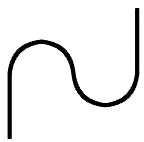
F () $:= c_{A,A} : A \otimes A \to A \otimes A; a \otimes b \mapsto a_1 b S(a_2) \otimes a_3$

F () $:= c^{-1}_{A,A} : A \otimes A \to A \otimes A; a \otimes b \mapsto a_3 \otimes S^{-1}(a_2) b a_1$

F () $: A \otimes A \to A; a \otimes b \mapsto ab$

A : unimoduler

$\Rightarrow$ • $\langle a, b \rangle := \lambda(ab)$ is a nondegenerate bilinear form

$\Rightarrow F$ () $= F$ ( ) $= F$ ()

• F (), F () are morphisms of ${}^A_A\mathcal{YD}$.

Def V : a Yetter–Drinfeld module

$\overset{\text{def}}{\Leftrightarrow}$ • V : a left A-module with $\triangleright : A \otimes V \to V; a \otimes v \mapsto a \triangleright v$

• V : a left A-comodule with $\rho : V \to A \otimes V; v \mapsto v_{-1} \otimes v_0$

• $\rho(a \triangleright v) = a_1 v_{-1} S(a_3) \otimes a_2 \triangleright v_0$

${}^A_A\mathcal{YD}$: the category of Yetter–Drinfeld modules

objects: Yetter–Drinfeld modules

morphisms: A-linear A-colinear maps

$A \in {}^A_A\mathcal{YD}$ with $a \triangleright v = a_1 v S(a_2)$ and $\rho(v) = \Delta(v)$

${}^A_A\mathcal{YD}$ is a braided tensor category with the braiding

$$c_{V,W} : V \otimes W \to W \otimes V; v \otimes w \mapsto v_{-1} \triangleright w \otimes v_0$$

$\boldsymbol{A_m}$: the Hopf algebra defined by

generators: x, y, z

relations: $x^2 = y^2 = 1$, $yx = xy$,

$zx = xz$, $z^m = x$, $zy = yz^{-1}$

$$\Delta(x) = x \otimes x$$

$$\Delta(y) = \frac{y\otimes(y+xy)+yz\otimes(y-xy)}{2}$$

$$\Delta(z) = \frac{z\otimes(z+xz)+z^{-1}\otimes(z-xz)}{2}$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$$

$$S(x) = x$$

$$S(y) = \frac{y+yx+yz-yxz}{2}$$

$$S(z) = \frac{z^{-1}+xz^{-1}+z-xz}{2}$$

m	2	3	4	5	6
0_1	64	144	256	400	576
4_1	64	216	256	400	864
5_1	64	144	256	400	576
5_2	64	216	256	400	864
5_3	64	144	256	400	576
5_4	64	144	256	400	576
6_1	64	144	256	400	576
6_2	64	144	256	400	576
6_3	64	144	256	400	576
6_4	64	144	256	400	576
6_5	64	144	256	400	576
6_6	64	144	256	400	576
6_7	64	144	256	800	576
6_8	64	144	256	400	576
6_9	64	216	256	400	864
6_{10}	64	144	256	400	576
6_{11}	64	144	256	400	576
6_{12}	64	144	256	800	576
6_{13}	64	216	256	400	864
6_{14}	64	288	256	400	1152
6_{15}	64	288	256	400	1152
6_{16}	64	144	256	400	576