Unstabilized Heegaard Splittings

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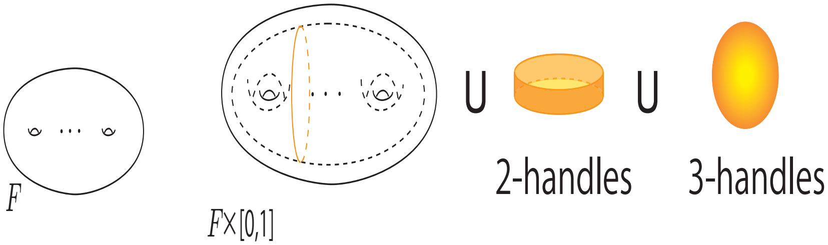
Heegaard Splitting I

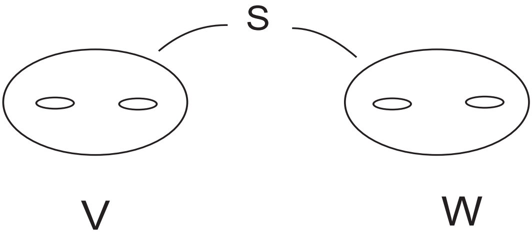
M : compact, orientable 3-manifold.

- Handlebody, Compression body.



Heegaard Splitting II

- Heegaard splitting : $M = V \cup_S W$, where V, W are compression bodies.



Theorem 1

M admits a Heegaard splitting.

Different Heegaard splittings

- $V_1 \cup_{S_1} W_1$, $V_2 \cup_{S_2} W_2$, Heegaard splittings of M.

- Isotopic Heegaard splittings. $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ are isotopic if
$\exists\ i : M \times [0, 1] \to M$, such that

$i \mid M \times \{0\}$ is identity,

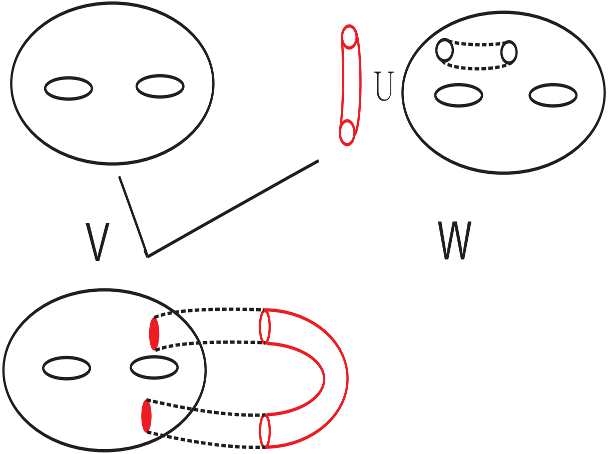
$i \mid M \times \{t\}$ is a homeomorphism,

$i \mid M \times \{1\}$ sends S_1 to S_2, V_1 to V_2 and W_1 to W_2.

- $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ are different if they are not isotopic.

Stabilization I

- Stabilization.



Stabilization II

- Stabilized Heegaard splitting, unstabilized Heegaard splitting.
- Reducible Heegaard splitting.

Fact 1

A stabilized Heegaard splitting is reducible or a genus 1 Heegaard splitting of S^3.

- A Heegaard splitting is different from its stabilization.

Heegaard distance

- $V \cup_S W$, a Heegaard splitting.
- e.s.c.c α, $\beta \subset S$.

$d(\alpha, \beta) = \min\{n \mid \alpha = \alpha_0, \alpha_1, ..., \alpha_n = \beta, \alpha_i \cap \alpha_{i+1} = \emptyset, \forall i \leq n-1\}$.

- Heegaard distance $d(S) = \min\{d(\alpha, \beta)\}$, where

$\alpha = \partial D$, essential disk $D \subset V$,

$\beta = \partial E$, essential disk $E \subset W$.

- If V or W is trivial, i.e., closed surface I-bundle, then define $d(S) = -\infty$.

For more detail, see [Hem], [Sch], [MM], [Schl].

Connect Sum

- F essential 2-sphere. $M = M_1 \# M_2$.

- Connect sum of two Heegaard splittings.

$M_1 = V_1 \cup_{S_1} W_1$, $M_2 = V_2 \cup_{S_2} W_2$.

V, disk sum of V_1 and V_2. W, disk sum of W_1 and W_2.

$V \cup_S W$, connect sum of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.

- D.Bachman and RF Qiu proved the following theorem 2 independently:

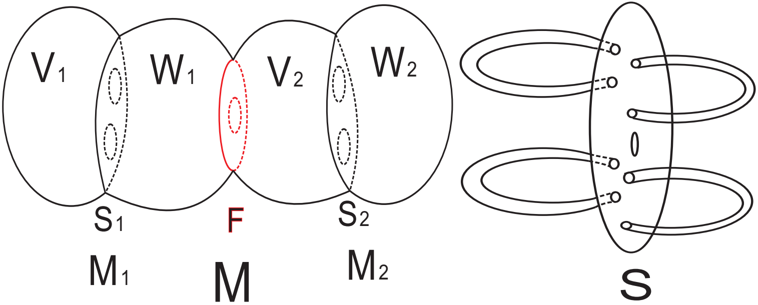
Theorem 2

The connect sum of any two unstabilized Heegaard splittings is unstabilized.

See [Ba] and [QS]

Amalgamation I

- From now on, we assume M is irreducible.
- $F \subset M$, a separating closed essential surface with $g(F) \geq 1$.
- $M = M_1 \cup_F M_2$. Amalgamation of two Heegaard splittings.



Amalgamation II

Question 1

Is the amalgamation of any two unstabilized Heegaard splittings unstabilized?

Answer: No.See [KQRW] and [SW].

However, Bachman-Schleimer-Sedgwick, Lackenby, Li and Souto' works show

Theorem 3

If the gluing map $f : F \to F$ is sufficiently complicated, then the amalgamation of any two minimal Heegaard splittings is unstabilized.

[BSS], [La], [Li1], [Li2], [So]

Amalgamation III

While, Kobayashi-Qiu, Yang-Lei prove

Theorem 4

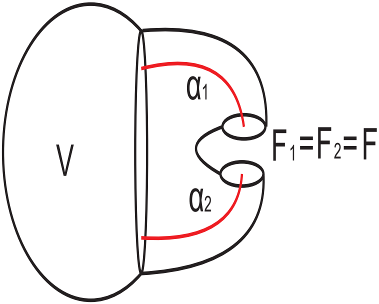
If both Heegaard distance $d(S_1)$ and $d(S_2)$ are large enough, then $V \cup_S W$ is unstabilized.

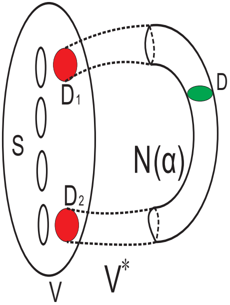
See [KQ], [YL].

Self-amalgamation I

- $F \subset M$, a non separating closed essential surface with $g(F) \geq 1$.
Then $M = M_1 \cup_f$.

- Self-amalgamation.





Self-amalgamation II

Question 2

Is the self amalgamation of an unstabilized Heegaard splittings unstabilized?

Answer: no! See the counter example in [DQ].

However, Du-Qiu prove

Theorem 5

If $d(S)$ is large enough, then the self amalgamation of the Heegaard splitting is unstabilized.

See [DQ].

Recently, we got a new bound.

Self-amalgamation III

Theorem 6

If $d(S) \geq 3$, then the self amalgamation of the Heegaard splitting is unstabilized.

See [ZDGQ].

Remark. Since the counter example in [DQ] is strongly irreducible, 3 is the best.

While, if $V \cup_S W$ is minimal, Guo-Zou prove

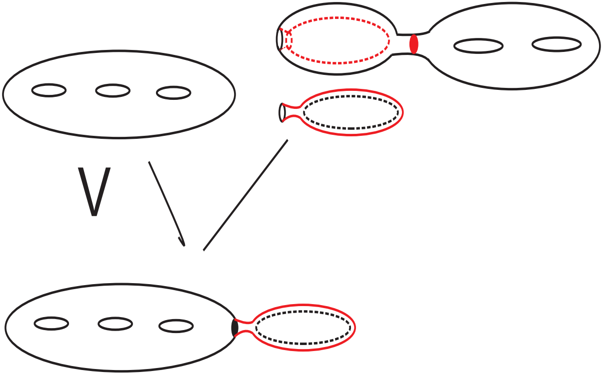
Theorem 7

If the gluing map $f : F \to F$ is complicated enough and M_1 is small, i.e. it contains no closed essential surface, then the self amalgamation of the Heegaard splitting is unstabilized.

See [GZ].

∂-stabilization I

- $V \cup_S W$, a Heegaard splitting. a closed surface $F \subset \partial_- W$.
- ∂-stabilization.



∂-stabilization II

Question 3

Is the ∂-stabilization of an unstabilized Heegaard splittings unstabilized?

Answer: no! See the counter example in [Mo].
But under some condition, We prove

Theorem 7

If $d(S) \geq 6$, then the ∂-stabilization of the Heegaard splitting is unstabilized.

See [ZGQ].

Remark. It means a 3-manifold M admitting high distance with boundary has two different unstabilized Heegaard splittings.

∂-stabilization III

Since the counter example in [Mo] has distance less than or equal to 2,

Conjecture

If $d(S) \geq 3$, then the ∂-stabilization of the Heegaard splitting is unstabilized.

谢谢！ Thank you!

ありがとう！ 감사 합니다！

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