On the distance of bridge spheres for knots

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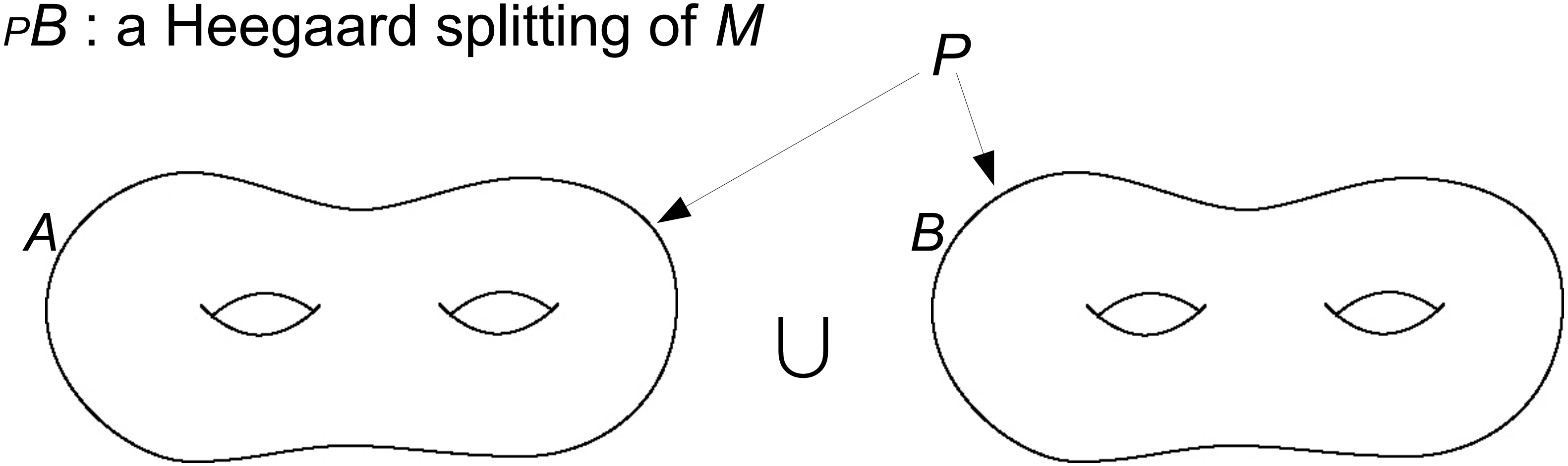
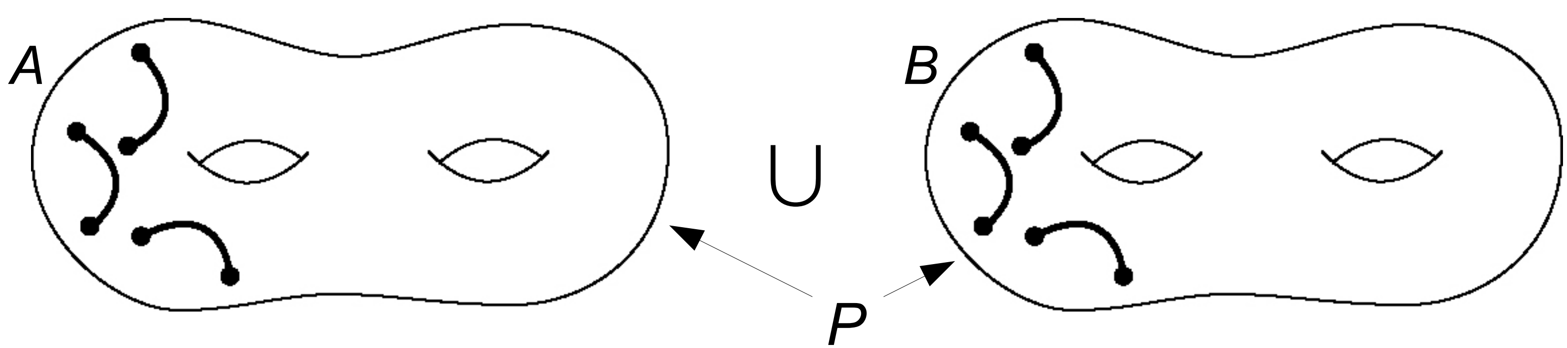
Definitions

M : a closed orientable irreducible 3-manifold
K : a knot in M
$A \cup_P B$: a Heegaard splitting of M



We say that K is in a (genus g) n-bridge position (with respect to $A \cup_P B$) if $K \cap A$ (resp. $K \cap B$) is a system of trivial n arcs in A (resp. B).



Then, P is called a bridge surface for K.

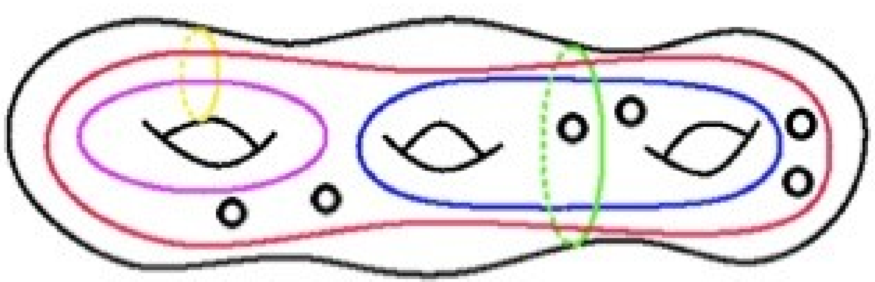
Definition (curve complex)

S : an orientable surface with genus≧2 (possibly with punctured)

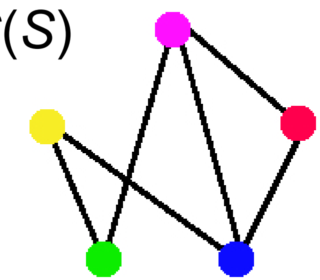
C(*S*) : the curve complex of *S* s.t.

- the vertices corresponding to isotopy classes of ess. s.c.c.'s on *S*.
- the edges are drawn between vertices corresponding to disjoint curves.

S



C(*S*)



The distance between two vertices is the smallest number of edges in any path between the vertices in *C*(*S*).

※ The curve complex for a non-punctured torus and a 4 punctured sphere are not connected.

Definition (Hempel distance)

M : a closed orientable irreducible 3-manifold with a H.S. $A \cup_P B$
K : a knot in M which is in bridge position with respect to P

$V(A)$ (resp. $V(B)$) : the subset of the curve complex $C(P \setminus K)$ corresponding to ess.s.c.c.'s on $P \setminus K$ bounding disks in $A \setminus K$ (resp. $B \setminus K$)

Then $d(P, K) := d(V(A), V(B))$

Motivations

Theorem [Hartshorn].

M : a closed orientable irreducible 3-manifold with Heegaard surface F
S : an orientable incompressible surface in M

$$d(F) \leqq 2g(S)$$

Theorem [Bachman-Schleimer]

M : a closed orientable irreducible 3-manifold with Heegaard surface F
K : a knot in M which is in bridge position with respect to F
S : a properly embedded, orientable, essential surface in M.

$$d(F, K) \leqq 2g(S) + |\partial(S \setminus K)|$$

K. Hartshorn. *Heegaard splittings of haken manifolds have bounded distance*.Pacific J. Math., 204(1): 61–75, 2002

D. Bachman and S. Schleimer. *Distance and bridge position.* Pacific J. Math., 219(2):221–235, 2005.

Theorem [Tomova]

M : a closed orientable irreducible 3-manifold

K : a non-trivial knot in M

P, Q : bridge surfaces for K that is not a 4-times punctured sphere.

$$P \not\simeq Q \Rightarrow d(K,P) \leq 2-\chi(Q \setminus K).$$

Corollary [Tomova]

If $K \subset S^3$ is in minimal bridge position w.r.t. a sphere P s.t. $d(K,P) > |P \cap K|$ then K has a unique minimal bridge position.

M. Tomova, *Multiple bridge surfaces restrict knot distance*, Algebr. Geom. Topol. 7 (2007), 957–1006.

Theorem [Tomova]

M : a closed orientable irreducible 3-manifold

K : a non-trivial knot in M

P, Q : bridge surfaces for K that is not a 4-times punctured sphere.

$$P \neq Q \Rightarrow d(K,P) \leq 2-\chi(Q \setminus K).$$

Corollary [Tomova]

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Main result

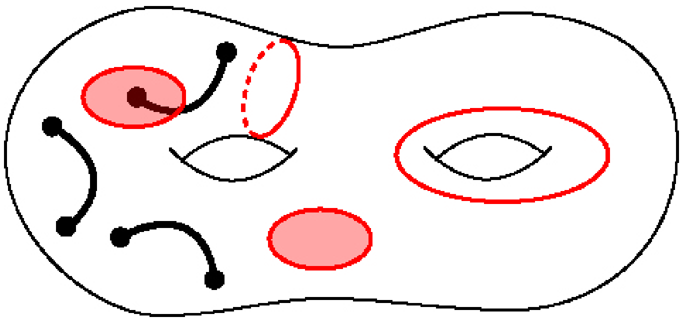
If $K \subset S^3$ is in minimal bridge position w.r.t. a sphere P s.t. $d(K,P) > |P \cap K|-2$ then K has a unique minimal bridge position

Fundamental settings

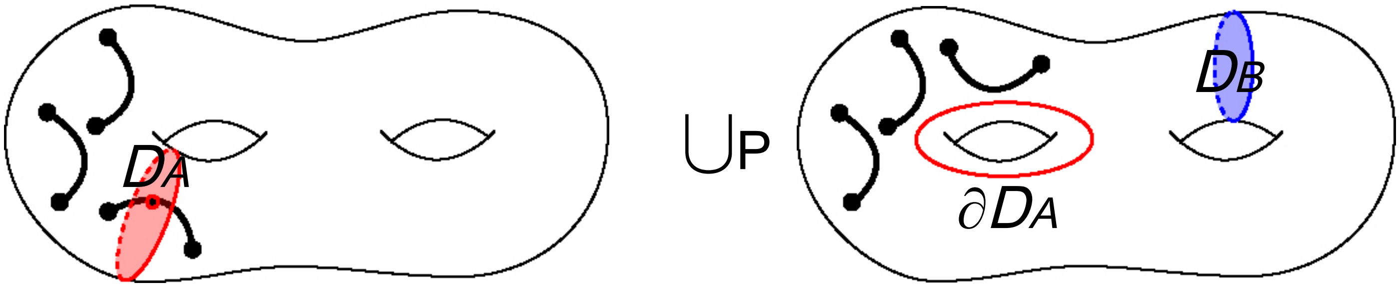
M : a closed orientable irreducible 3-manifold with Heegaard splitting $A \cup_P B$
K : a knot in *M* which is in bridge position with respect to $A \cup_P B$

Definition
A surface *D* in *M* is a K-disk, if *D* is a disk intersecting *K* in at most one transverse point.

Definition
ℓ : a s.c.c in *P*
ℓ is K-inessential if ℓ bounds a K-disk in *P*.
ℓ is K-essential if it is not K-inessential.



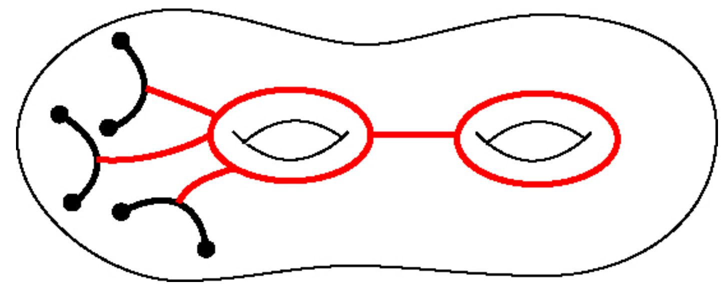
Definition
We say that $A \cup_P B$ is weakly K-reducible if there exist K-disks D_A, D_B in *A, B* respectively such that ∂D_A, ∂D_B are K-essential on *P* and $\partial D_A \cap \partial D_B = \Phi$.



$A \cup_P B$ is strongly K-irreducible if it is not weakly K-reducible.

M : a closed orientable irreducible 3-manifold with Heegaard splitting $A \cup_P B$
K : a knot in M which is in bridge position with respect to $A \cup_P B$
Σ_A (resp. Σ_B) : a spine of (A, K) (resp. (B, K))

(Hence $M \backslash (\Sigma_A \cup \Sigma_B) \simeq P \times (0,1)$)

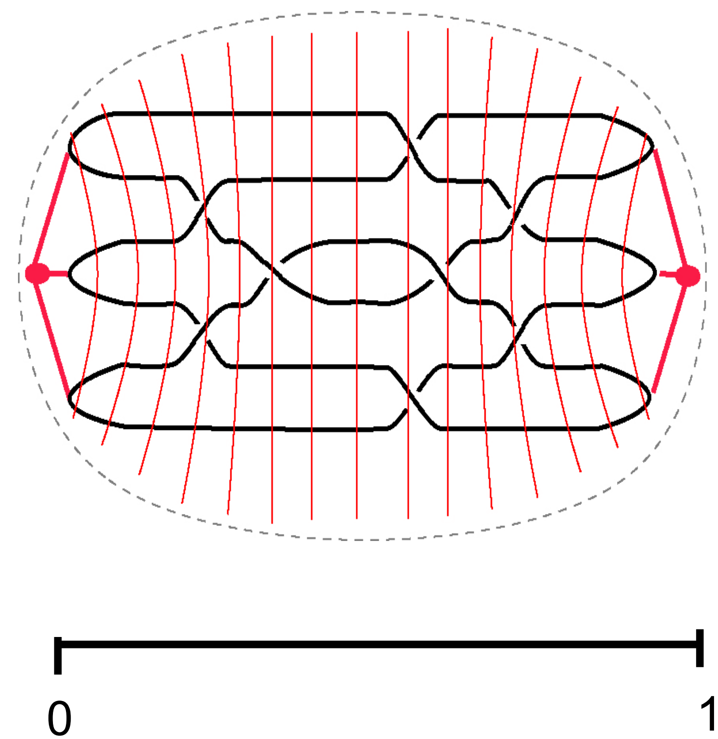


Definition (sweep-out)

We may suppose $M \backslash (\Sigma_A \cup \Sigma_B) \simeq P \times (0,1) \to (0,1)$ extends to a smooth map

$f : M \to [0, 1]$
s.t. • $f^{-1}(0) = \Sigma_A$
• $f^{-1}(1) = \Sigma_B$
• $f^{-1}(1/2) = P$

Regard f as a height function
(:called sweep-out)



Notation
$P_t := f^{-1}(t)$ ($0 < t < 1$)
$A_t := f^{-1}([0,t])$, $B_t := f^{-1}([t,1])$

Outline of the proof

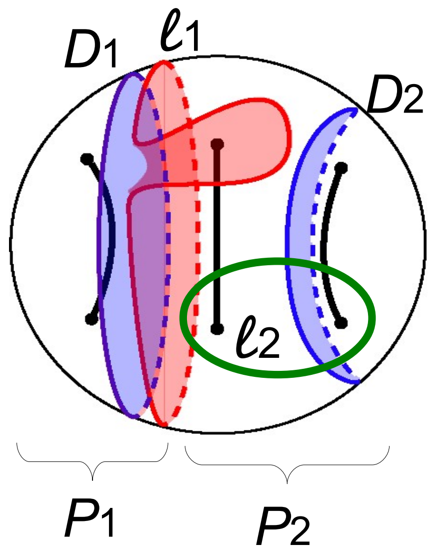
$A \cup_P B$, $X \cup_Q Y$: genus 0 Heegaard splittings for S^3
K : a knot in S^3 s.t. K is in n-bridge position w. r. t. P, Q

Lemma 1

ℓ_1 : a K-ess. s.c.c. bounding K-disk in A
ℓ_2 : a K-ess. s.c.c. on P s.t. $\ell_1 \cap \ell_2 = \Phi$

$\Rightarrow d(V(A), \ell_2) \leqq 1$

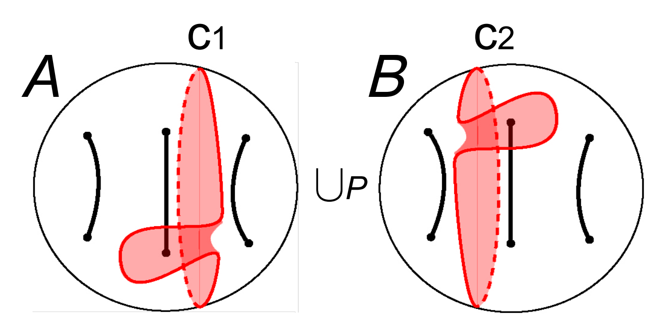
$\because \ell_1$ separate P two comp. P_1, P_2.
$\exists D_1$: a K-disk s.t. ∂D_1 is K-ess. on P_1
(resp. D_2) (resp. ∂D_2) (resp. P_2)
$\ell_2 \subset P_1 \Rightarrow \ell_2 \cap \partial D_2 = \Phi$, $\ell_2 \subset P_2 \Rightarrow \ell_2 \cap \partial D_1 = \Phi$



Lemma 2

P : weakly K-red. $\Rightarrow d(P, K) \leqq 3$

$\because d(P,K) \leqq d(V(A), c_1) + d(c_1, c_2) + d(c_2, V(B)) = 3$

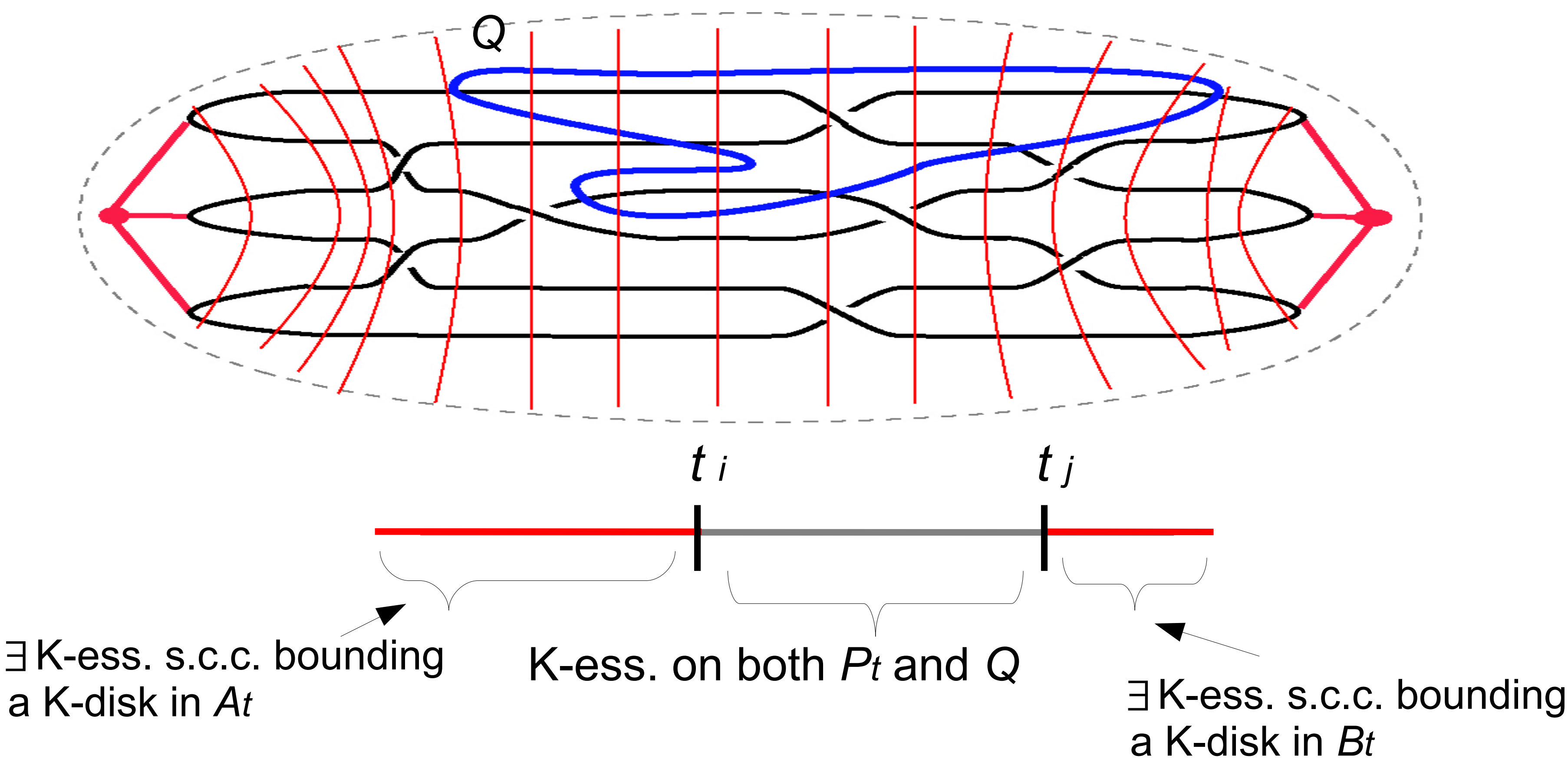


Lemma 3

P, Q : strongly K-irred.

$P \not\simeq Q \Rightarrow \exists f : M \to [0,1]$: a sweep-out induced from P s.t. by isotopy, for $P_t \cap Q$ satisfys the following : $\exists (t_i, t_j) \subset [0,1]$ s.t.

- for $(t_i, t_j) \subset [0,1]$, each comp. of $P_t \cap Q$ is K-ess. on both P_t and Q.
- for any small $\varepsilon > 0$, $P_{t_i - \varepsilon} \cap Q$ contains a K-ess. s.c.c bounding a K-disk in A_t
- for any small $\varepsilon > 0$, $P_{t_j + \varepsilon} \cap Q$ contains a K-ess. s.c.c bounding a K-disk in B_t



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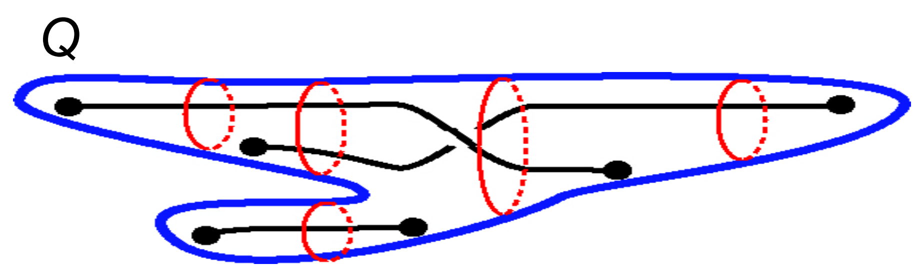
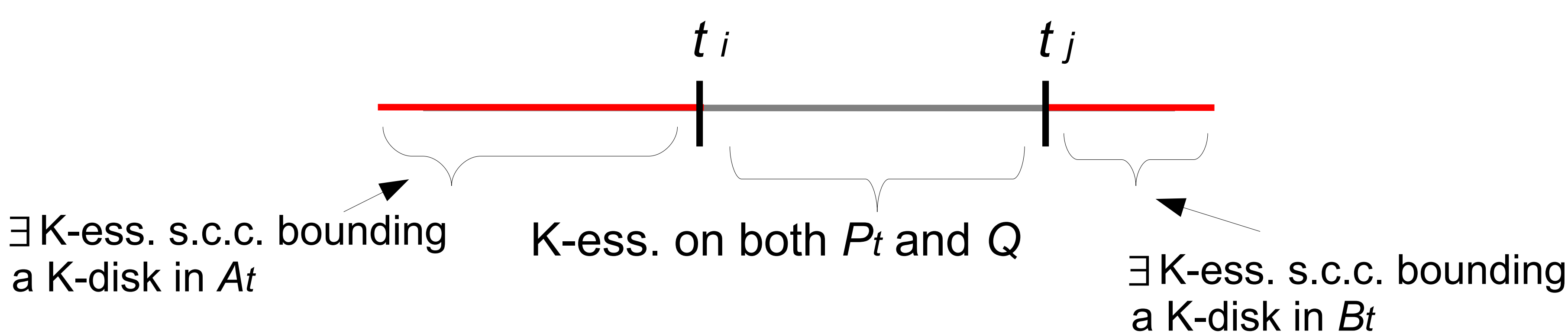
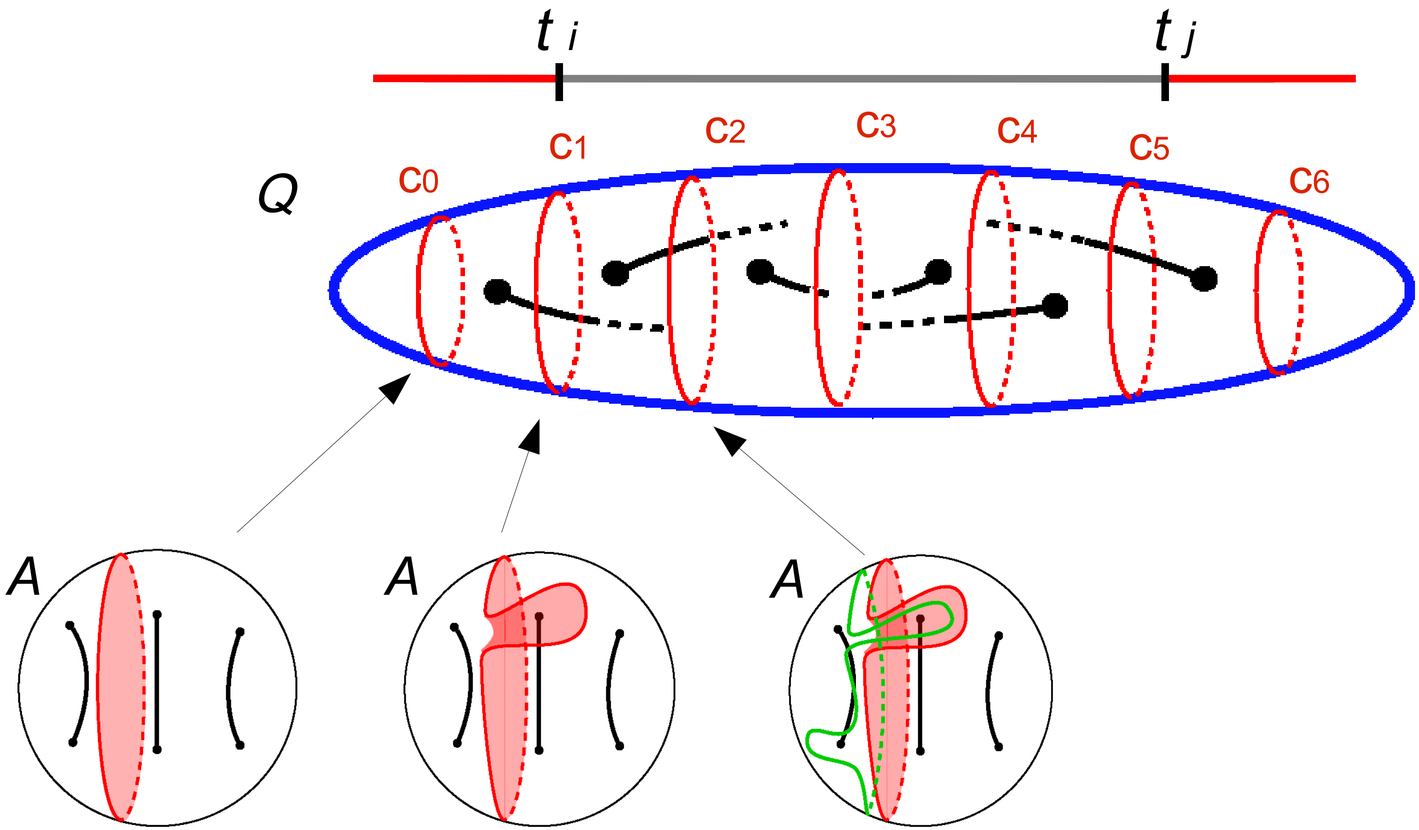




Suppose $d(P,K)=|P\cap K|$

Then we can isotope Q so that :



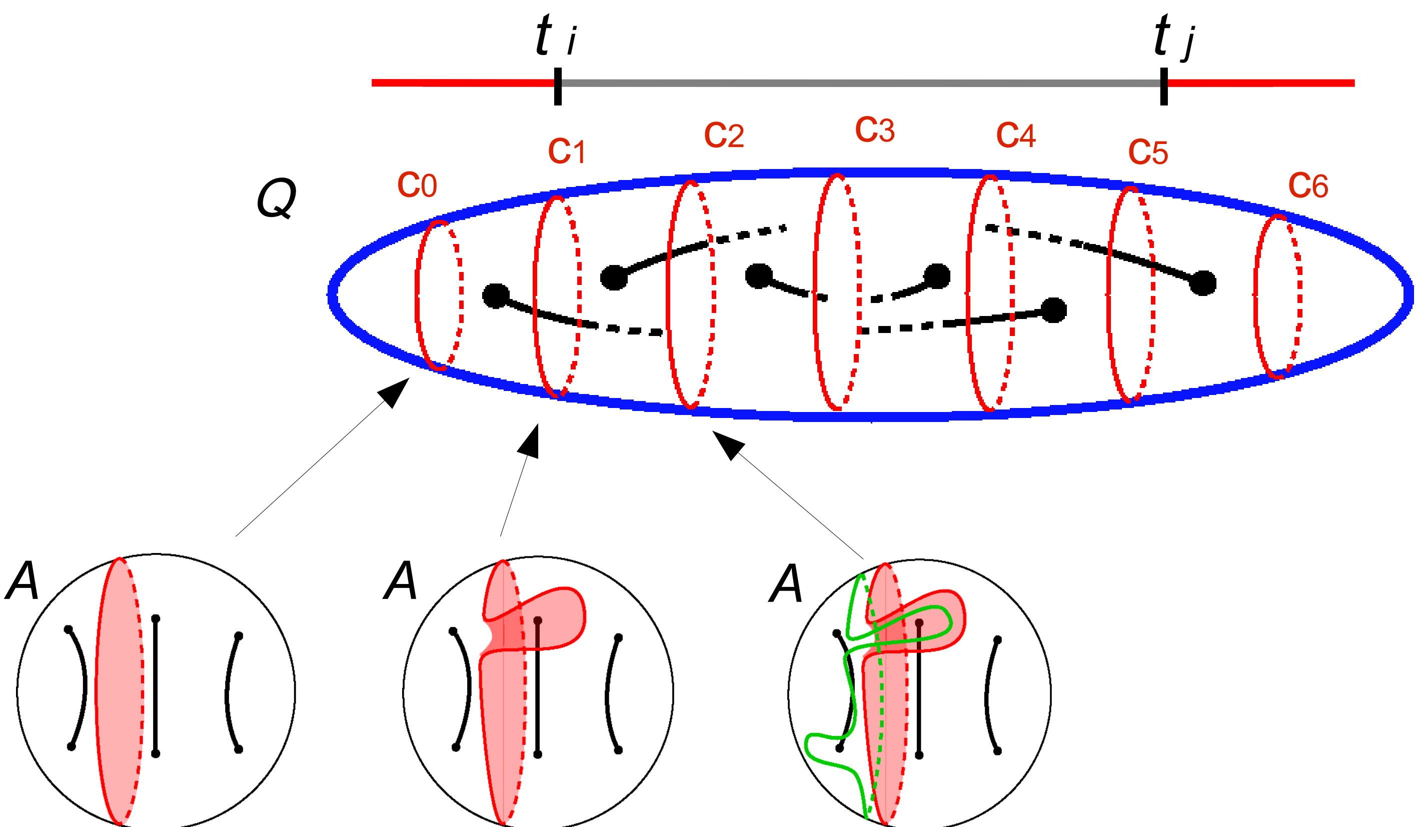
$d(V(A),c_0)=0$

$d(V(A),c_1)\leqq 1$

$d(c_1,c_2)\leqq 1$,
by Lemma 1,
$d(V(A),c_2)\leqq 1$

Suppose d(P,K)=|P∩K|

Then we can isotope Q so that :



d(V(A),c_0)=0

d(V(A),c_1)≦1

d(c_1,c_2)≦1,
by Lemma 1,
d(V(A),c_2)≦1

∴d(K,P)≦|P∩K|-2

Similarly we can show the case of d(K,P)=|P∩K|-1. □

Thank you