On symmetry-structural links and the loop space of S^3

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For S^3, it is known that ΩS^3 has a good model $J(S^2)$, i.e., $J(S^2) \simeq \Omega S^3$

Definition of symmetry-structural links

An n-link L in S^3 is called to be *symmetry-structural* if any two (oriented) k-sublinks of L are equivalent to each other up to ambient isotopy for $1 \leq k \leq n$.

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Precisely, let $L = \{l_1, \ldots, l_n\}$ be an n-link, where l_i is the ith component of L. Then L is symmetry-structural if and only if for any two sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ the sub links

$$\{l_{i_1}, l_{i_2}, \ldots, l_{i_k}\} \sim \{l_{j_1}, l_{j_2}, \ldots, l_{j_k}\}$$

in the sense that there exists an isotopy $H_t \colon S^3 \to S^3$ with $H_0 = \mathrm{id}$ and $H_1(l_{i_s}) = l_{j_s}$ for $1 \leq s \leq k$.

Examples of symmetry-structural links

Consider the Hopf fibration $p\colon S^3 \longrightarrow S^2$. Let $Q_n = \{q_1, \dots, q_n\} \subseteq S^2$ be the n distinct points in S^2. Let $L_n = p^{-1}(Q_n)$. Then $L_n = \{l_1, \dots, l_n\}$ is an n-link in S^3, where $l_i = p^{-1}(q_i)$. L_n is the Hopf link with n components.

Clearly, L_n is a symmetry-structural link.

Examples of symmetry-structural links

Let K be a tame knot with an extension of an embedding $\phi\colon D^2 \times S^1 \hookrightarrow S^3$. Take a ribbon $\phi|\colon [0,1] \times S^1 \hookrightarrow D^2 \times S^1 \hookrightarrow S^3$. Given $0 = t_0 < t_1 < \cdots < t_n < 1$, the $(n+1)$-link $L = \{l_0, \ldots, l_n\}$ with $l_i = \phi(\{t_i\} \times S^1)$ is symmetry-structural with $l_0 = K$.

In other words, any knot can be a component of a symmetry-structural link.

Δ-group

Recall that a Δ-group means a sequence of groups $G = \{G_n\}_{n\geq 0}$ with faces $d_i : G_n \to G_{n-1}$, $0 \leq i \leq n$, such that

$$d_i d_j = d_j d_{i+1}$$

for $i \geq j$, which is called the Δ-identity.

Simplicial group

A simplicial group means a Δ-group $G = \{G_n\}_{n\geq 0}$ together with a collection of degeneracies $s_i : G_n \to G_{n+1}$, $0 \leq i \leq n$, such that

$$d_j d_i = d_{i-1} d_j$$

for $j < i$,

$$s_j s_i = s_{i+1} s_j$$

for $j \geq i$ and

$$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ id & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases}$$

The three identities for $d_j d_i$, $s_j s_i$ and $d_j s_i$ are called the simplicial identities.

A Δ-group $G = \{G_j\}_{0 \leq j \leq n}$ is called a partial simplicial group if there exist degeneracies $s_k : X_j \to X_{j+1}$, $0 \leq k \leq j$, such that the simplicial identities hold.

For any simplicial group $G = \{G_j\}_{0 \leq j < \infty}$, the truncated piece $\{G_j\}_{0 \leq j \leq n}$ is a partial simplicial group.

the Δ-structure on symmetry-structural links

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Set $G(L_n) = \pi_1(S^3 \backslash L_n)$, the fundamental group of the link complement.

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Set $G(L_n) = \pi_1(S^3 \backslash L_n)$, the fundamental group of the link complement.

Since $d_j L \subset L$, the inclusion $S^3 \backslash L \hookrightarrow S^3 \backslash d_j L$ induces a homomorphism

$$d_j : G(S^3 \backslash L) \to G(S^3 \backslash d_j L).$$

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Similarly, we have

$$s_i : G(S^3 \backslash s_i L) \to G(S^3 \backslash L).$$

Given a symmetry-structural $(n+1)$-link L_n, define a sequence of links $\mathbb{L} = \{L_j\}_{0 \leq j \leq n}$, here L_j is a sub $(j+1)$-link of L_n.

Since L_n is symmetry-structural, each L_j is unique up to ambient isotopy. Thus we can choose

$$\begin{aligned} L_{n-1} &= d_1 L_n, \\ L_{n-2} &= d_1^2 L_n = d_1(d_1 L_n), \\ &\cdots, \\ L_0 &= d_1^n L_n. \end{aligned}$$

Now we consider the sequence of link groups $\{G(L_j)\}_{0\leq j\leq n}$.

The inclusion of link complements

$$S^3\backslash L_j \hookrightarrow S^3\backslash d_k L_j \sim S^3\backslash L_{j-1}$$

induces a group homomorphism, which we also denote it by d_k,

$$d_k : G(L_j) \longrightarrow G(L_{j-1}), 0 \leq k \leq j.$$

Hence we get the face operations

$$d_k : G(L_j) \longrightarrow G(L_{j-1})$$

for $0 \leq k \leq j$.

Proposition

Suppose L_n is a symmetry-structural $(n+1)$-link in S^3, then $G(\mathbb{L}) = \{G(L_j)\}_{0 \leq j \leq n}$ together with $d_k : G(L_j) \longrightarrow G(L_{j-1})$, $0 \leq k \leq j$ forms a Δ-group.

Satellite symmetry-structural links

Let $L_n = \{l_0, l_1, \cdots, l_n\}$ be an $(n+1)$-link in S^3. We call L_n a *satellite symmetry-structural link* if

1. L_n is symmetry-structural; and
2. For any $i = 0, 1, \cdots, n-1$, l_{i+1} is parallel to l_i under isotopy, that is, l_{i+1} cobounds a band with l_i.

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For a satellite symmetry-structural link L_n in S^3, we can define degeneracies $s_i : G(L_j) \longrightarrow G(L_{j+1})$, $0 \leq i \leq j$, as follows.

Satellite symmetry-structural links

Consider the following inclusions

$$S^3 \smallsetminus V_{2\delta}(L_j) \hookrightarrow S^3 \smallsetminus V_\delta(s_i L_j) \hookrightarrow S^3 \smallsetminus V_\delta(L_j) \hookrightarrow S^3 \smallsetminus V_{\frac{\delta}{2}}(L_{j+1}),$$

while

$$S^3 \smallsetminus L_j \sim S^3 \smallsetminus V_{2\delta}(L_j)$$

$$S^3 \smallsetminus L_{j+1} \sim S^3 \smallsetminus V_{\frac{\delta}{2}}(L_{j+1})$$

then we get the inclusion $S^3 \smallsetminus L_j \hookrightarrow S^3 \smallsetminus L_{j+1}$. This inclusion induces a group homomorphism, which we denote s_i,

$$s_i : G(L_j) \longrightarrow G(L_{j+1})$$

for $0 \leq i \leq j$.

Satellite symmetry-structural links

Proposition

Suppose L_n is a satellite symmetry-structural $(n+1)$-links in S^3, then $G = \{G(L_j)\}_{0 \le j \le n}$ together with $d_k : G(L_j) \longrightarrow G(L_{j-1})$, $0 \le k \le j$ and $s_i : G(L_j) \longrightarrow G(L_{j+1})$, $0 \le i \le j$ forms a partial simplicial group.

simplicial-structural links

We call a sequence of $(n+1)$-links $\mathbb{L} = \{L_n\}_{n\geq 0}$ a simplicial-structural link if

1. Each L_n is satellite symmetry-structural.
2. For any $n_1 < n_2$, L_{n_1} is ambient isotopic to a sublink of L_{n_2}.

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Proposition

Suppose $\mathbb{L} = \{L_n\}_{n\geq 0}$ *is a simplicial-structural link in* S^3, *then* $G(\mathbb{L}) = \{G(L_n)\}_{n\geq 0}$ *is a simplicial group.*

Main Theorem

Theorem (Lei-Li-Wu)

For a nontrivial simplicial-structural link $\mathbb{L}$ *in* S^3, $G(\mathbb{L}) \simeq \Omega S^3$. *That is, the canonical classifying space of* $G(\mathbb{L}) = \{G(L_n)\}_{n\geq 0}$ *has the same homotopy type with* S^3.

Main Theorem

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Remark

The above theorem establishes a model of link groups for the loop space of S^3, which is combinatorial and related to link groups.

Outline proof

Let BG be the classifying space of $G(\mathbb{L})$, where $\mathbb{L}$ is a nontrivial simplicial-structural link in S^3.

Goal: $BG \simeq S^3$.

For a space X, if

- $$\pi_1(X) = 0$$
- $$H_q(X) = \begin{cases} \mathbb{Z} & q = 0, 3 \\ 0 & q \neq 0, 3 \end{cases}$$

then by Hurewicz Theorem and Whitehead Theorem, $X \simeq S^3$.

So our goal change to compute

- $\pi_1(BG) = 0$
- $H_*(BG)$

We denote $X = BG$, then $\pi_1(X) = \pi_0(\Omega X) = \pi_0(G)$, here $G = G(\mathbb{L})$ is the link group, and it is a simplicial group.

We use the Moore chain complex.

Let G be a simplicial group. Define

$$N_nG = \bigcap_{j=1}^{n} Ker(d_j : G_n \to G_{n-1}).$$

The *Moore chain complex* NG is the sequence of groups

$$\cdots \longrightarrow N_3G \xrightarrow{d_0|} N_2G \xrightarrow{d_0|} N_1G \xrightarrow{d_0|} N_0G = G_0.$$

$$\pi_n(G) = Ker(N_nG \xrightarrow{d_0|} N_{n-1}G)/d_0(N_{n+1}G)$$

$\pi_0(G) = G_0/d_0(N_1G)$, where $N_1G = Ker(d_1 : G_1 \longrightarrow G_0)$.

We show that $N_1G \xrightarrow{d_0|} G_0$ is onto.

So $\pi_1(BG) = \pi_0(G) = 0$.

Then we compute $H_*(BG)$.

By using Quillen's spectral sequence, we have

$$E^2_{p,q} = \pi_p(\pi_q(\mathbb{Z}(BG_*)_*)) \Rightarrow \pi_{p+q}(\mathbb{Z}(BG_*)).$$

the E^1-terms are as follows:

1. $E^1_{0,*} \cong \mathbb{Z}(*)$
2. $E^1_{1,*} \cong \mathbb{Z}[\Delta[1]]$
3. $E^1_{2,*} \cong \mathbb{Z}[S^1]$
4. $E^1_{p,*} = 0, \ p > 2$

The E^2-terms are

$$E^2_{p,q} \cong \begin{cases} \mathbb{Z}, & p = q = 0 \\ \mathbb{Z}, & p = 2, q = 1 \\ 0, & \text{otherwise} \end{cases}$$

From Quillen's spectral sequence,

$$H_i(BG;\mathbb{Z}) = \pi_i(\mathbb{Z}(BG)) = \begin{cases} \mathbb{Z} & i = 0,3 \\ 0 & \textit{otherwise}. \end{cases}$$

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This completes the proof.

Thanks for your attention!