# Four-manifolds admitting hyperelliptic broken Lefschetz fibrations

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## $\diamondsuit$ Main Results (Roughly)

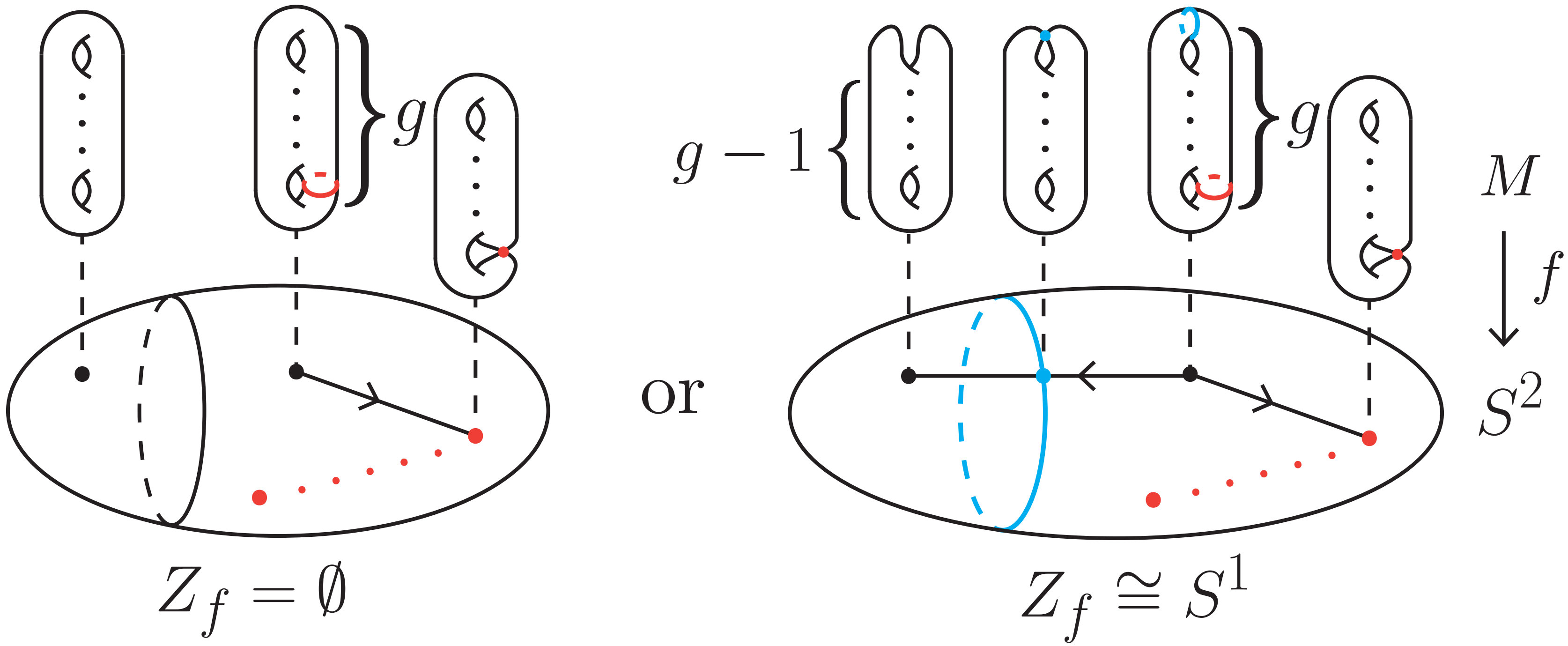
$f : M^4 \to S^2$: "hyperelliptic" "simplified broken Lefschetz fibration" w/ genus-$g$

**Theorem A.** $^\exists \omega : M \to M$: involution when $g \geq 3$

**Theorem B.** signature $\sigma(M)$ is determined by information of neighborhoods of singular fibers

$f : M \to S^2$ :**simplified broken Lefschetz fibration** (**SBLF**)

w/ genus-$g$




- $(z_1, z_2) \mapsto z_1 z_2$ (**Lefschetz singularity**),
- $(t, x_1, x_2, x_3) \mapsto (t, x_1^2 + x_2^2 - x_3^2)$ (**indefinite fold**, $Z_f$).

$*$ $f$ is a **Lefschetz fibration** if $f$ has no indefinite fold.

**Definition 1**

$f : M \to S^2$ is called a **simplified broken Lefschetz fibration** (**SBLF**) if:

- $f$ has at most the following two types of sing.
  - $(z_1, z_2) \mapsto z_1 z_2$ (**Lefschetz singularity**),
  - $(t, x_1, x_2, x_3) \mapsto (t, x_1^2 + x_2^2 - x_3^2)$ (**indefinite fold**, $Z_f$).
- $Z_f$ is either empty or embedded $S^1$ in $M$.
- $f|_{\text{(set of singularities)}}$ is injective.
- all the fibers of $f$ are connected.
- all the Lef. sing. are contained in the higher genus side.

## ◇ Background

**Donaldson 1999, Gompf 2004**

[symplectic $4$-manifold] $\Longleftrightarrow$ [Lefschetz pencil] $\xrightarrow{\text{blow up}}$ [Lefschetz fibration]

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**Auroux-Donaldson-Katzarkov generalized in 2005**

| near-symplectic 4-manifold | $\Longleftrightarrow$ | **BLP** with $^{\exists}$**condition** | $\xrightarrow{\text{blow up}}$ | **BLF** with $^{\exists}$**condition** |
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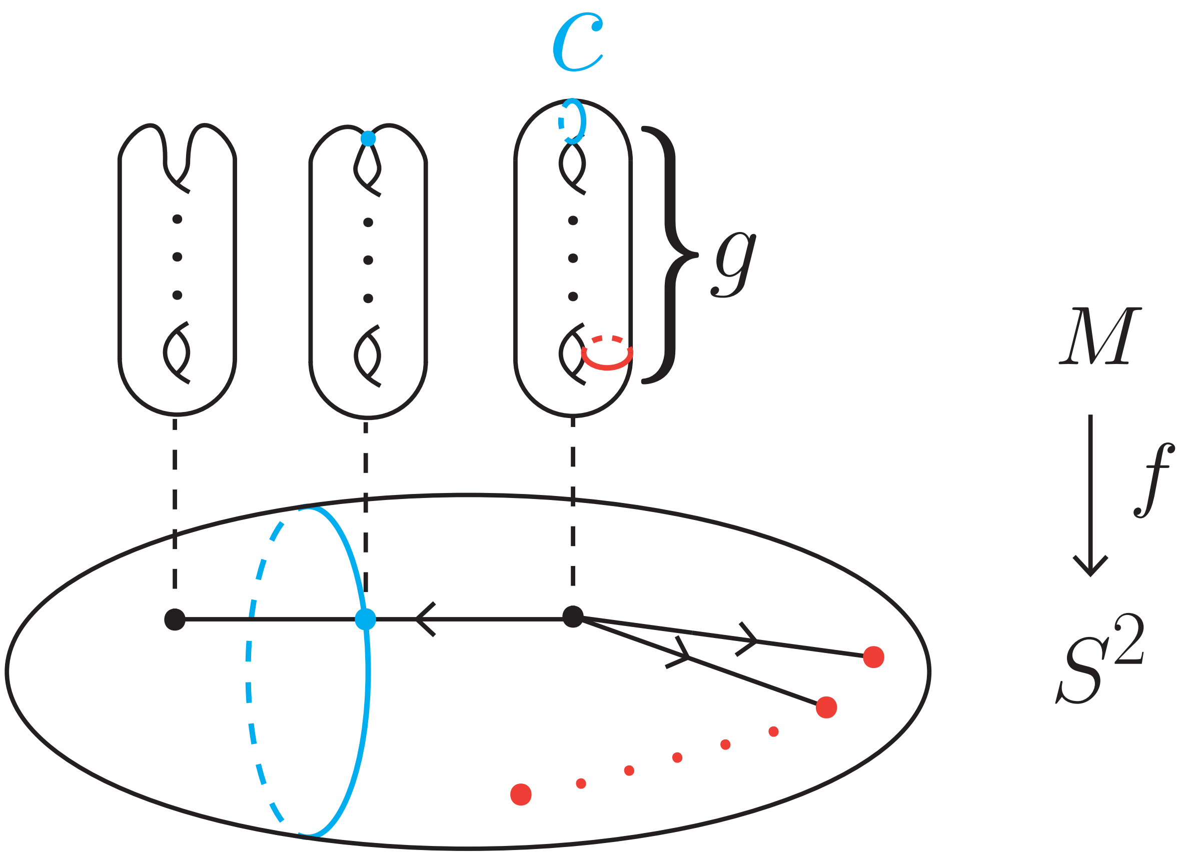
$^{\exists}$**BLF without a condition. Indeed,**

**Theorem 1 (Williams 2010 e.t.c.)**
$^{\forall}$closed ori. $4$-mfd admits a (**simplified**) **BLF**

## ◇ Vanishing cycles




$c$ is called a **vanishing cycle** of indefinite fold.

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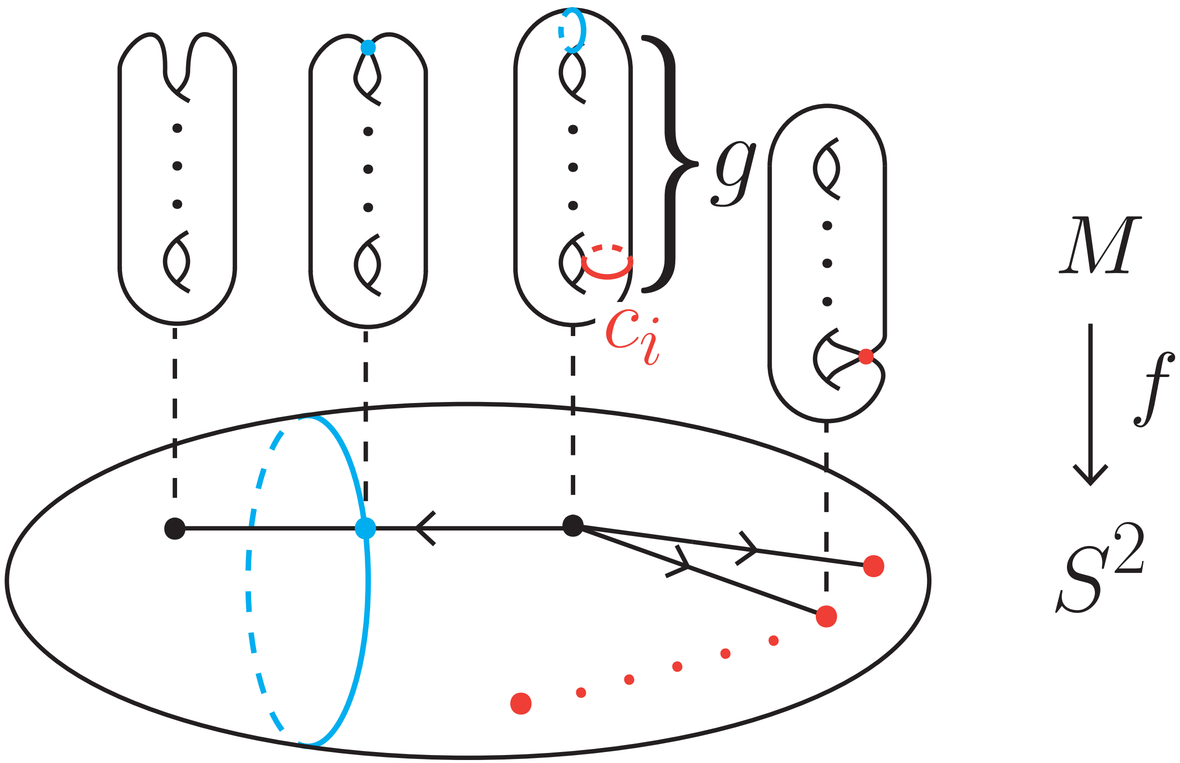
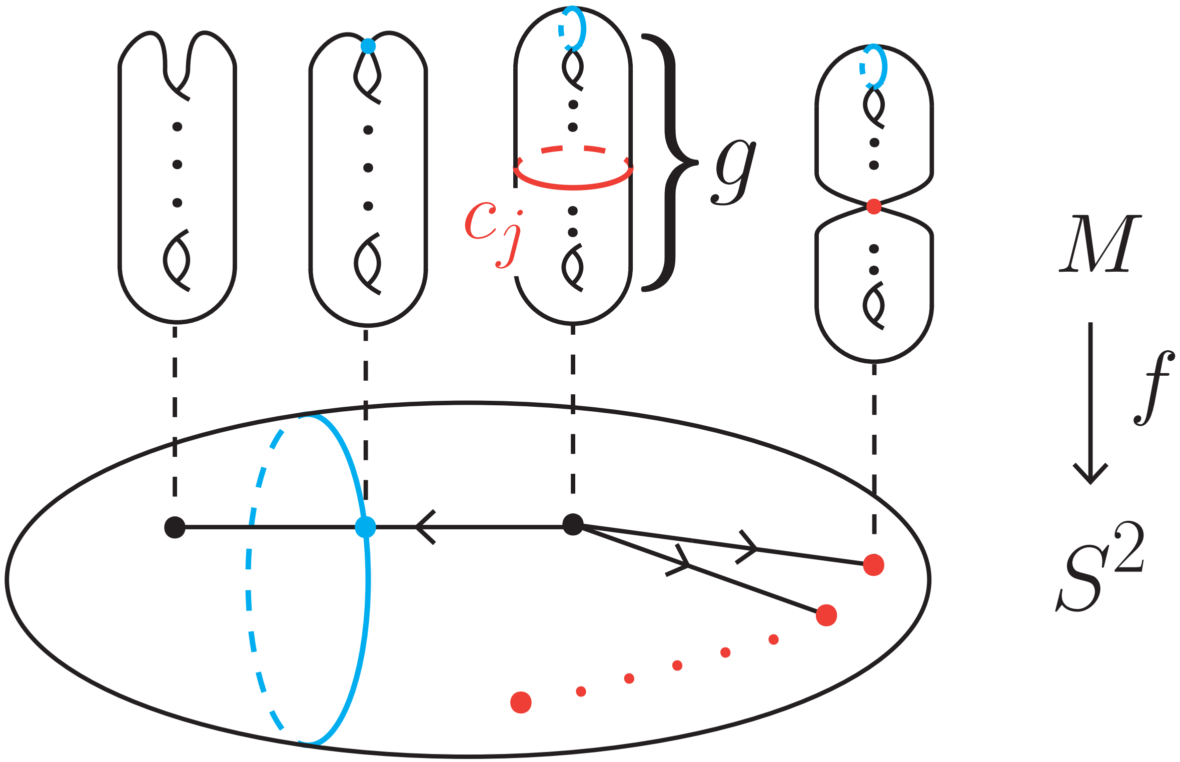



$\boldsymbol{c_i}$ is called a **vanishing cycle** of Lefschetz singularity.

## ◇ Vanishing cycles

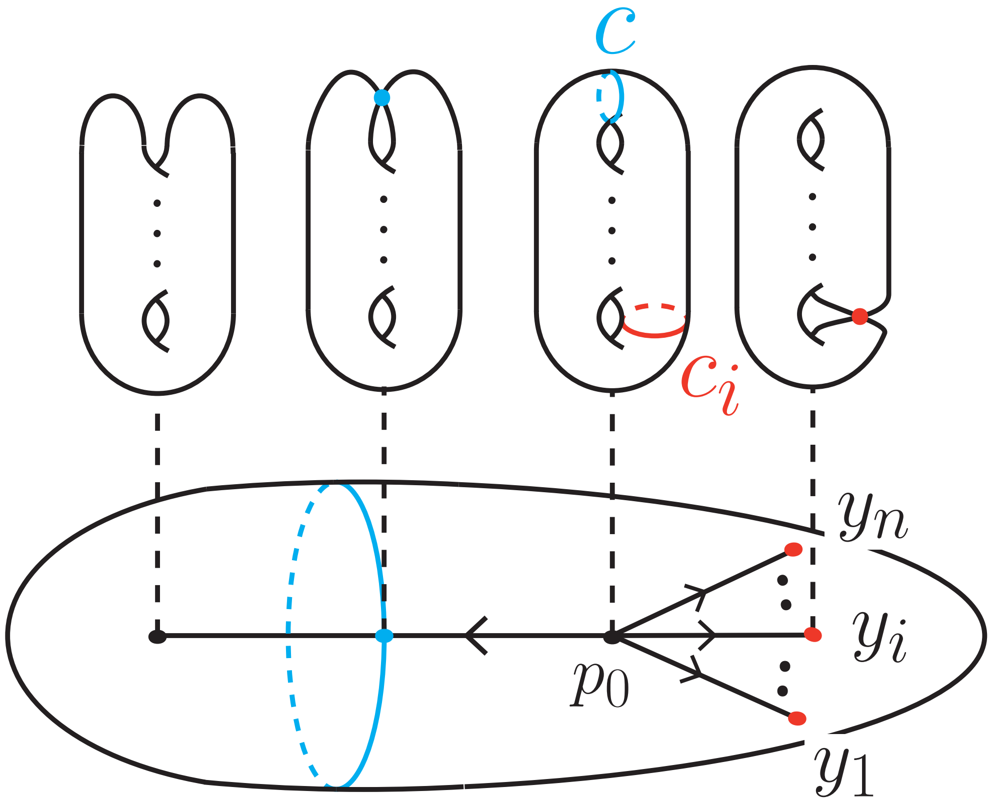



∗ van. cyc. of Lef. sing. can be separating.

## ◇ Hyperelliptic SBLF

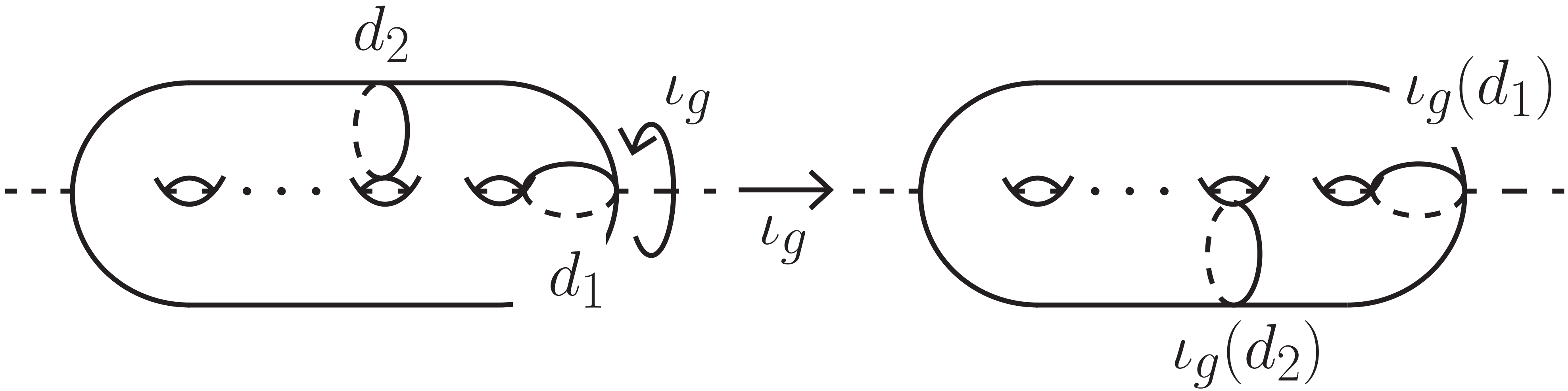
$f : M \to S^2$ :SBLF w/ genus-$g$, $f$(Lef. sing.) $= \{y_1, \ldots, y_n\}$

$c_1, \ldots, c_n, c$ :vanishing cycles of $f$



**Definition 2**

$f : M \to S^2$ :SBLF is said to be **hyperelliptic** (**HSBLF**) if $c_1, \ldots, c_n, c$ are preserved by the involution $\iota_g$ (up to isotopy).




* $\iota_g : \Sigma_g \to \Sigma_g$ is called a **hyperelliptic involution**.

* $^\forall$genus-$1$ or $2$ SBLF is hyperelliptic.

**Theorem A. (Sato-H.)**

$f : M \to S^2$ :HSBLF with genus-$g \geq 3$

(1) $s = \sharp\{$Lef. sing. with separating van. cycle$\}$

$M$ admits an involution $\omega : M \to M$ with

$M^{\omega} =$ ($2$-dim. part) $\amalg$ $\{s$ pts$\}$

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$\tilde{\omega} : M \# s\overline{\mathbb{CP}^2} \to M \# s\overline{\mathbb{CP}^2}$ : involution

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(2) $F \subset M$ a regular fiber of $f \Longrightarrow [F] \neq 0$ in $H_2(M;\mathbb{Q})$.

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(1) is a generalization of Siebert-Tian ('99) and Fuller ('00)'s results on HLFs.

**Corollary 1**

A definite $4$-manifold ($S^4, \#k\mathbb{CP}^2, \#k\overline{\mathbb{CP}^2}$ e.t.c.) cannot admit HSBLF w/ genus-$g \geq 3$.

$*$ $S^4, \#k\mathbb{CP}^2$ and $\#k\overline{\mathbb{CP}^2}$ admit SBLFs w/ genus-$^{\forall}g \geq 3$ but they would not be hyperelliptic.

**Corollary 2**

$f : M \to S^2$ :HSBLF with genus-$g \geq 3$

$^{\exists}\theta \in \Omega^2(M)$ :near symplectic form compatible with $f$.

$f : M \to S^2$ :SBLF with genus-$g$, $c, c_1, \ldots, c_n$: van. cycles.

**Lemma 1 (ADK '05)**

$t_{c_1} \cdot \cdots \cdot t_{c_n} \in \mathcal{M}_g(c)$,

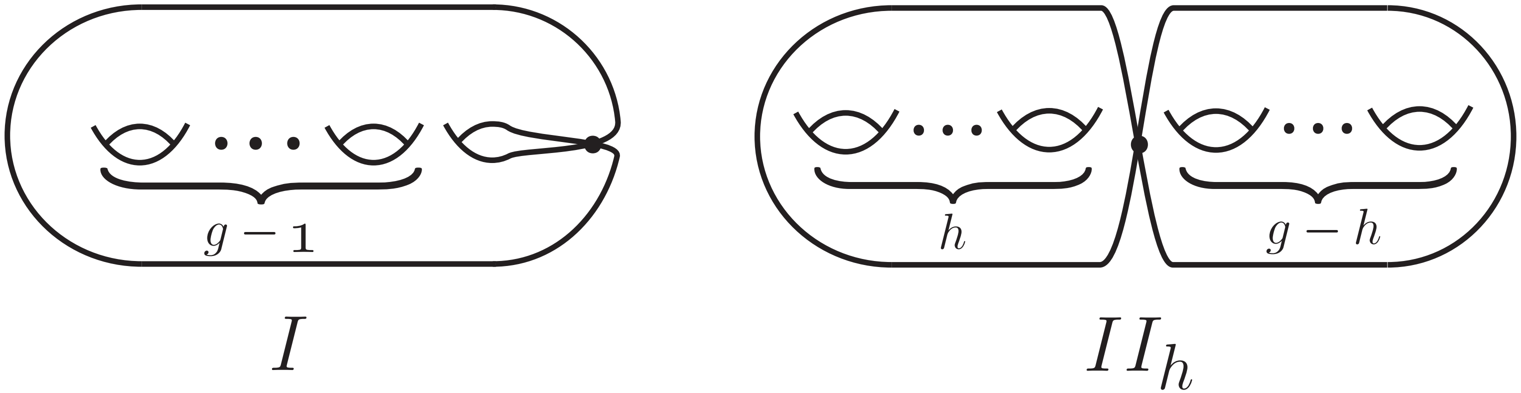
where $\mathcal{M}_g = \mathrm{Diff}^+ \Sigma_g$/isotopy,

$t_{c_i} \in \mathcal{M}_g$ is the right-handed Dehn twist along $c_i$ and

$\mathcal{M}_g(c) = \{[\varphi] \in \mathcal{M}_g | \varphi(c) = c\}$.

$*$ When $f$: hyperelliptic, $t_{c_1} \cdot \cdots \cdot t_{c_n} \in \mathcal{H}_g(c) = \mathcal{M}_g(c) \cap \mathcal{H}_g$,

where $\mathcal{H}_g = \{[\varphi] \in \mathcal{M}_g | \varphi \circ \iota_g = \iota_g \circ \varphi\}$




Assign rational numbers to the above singular fibers as

$$\boldsymbol{\sigma}_{\mathrm{loc}}(\boldsymbol{I}) = -\frac{\boldsymbol{g}+\mathbf{1}}{\mathbf{2}\boldsymbol{g}+\mathbf{1}}, \quad \boldsymbol{\sigma}_{\mathrm{loc}}(\boldsymbol{II_h}) = \frac{\mathbf{4}\boldsymbol{h}(\boldsymbol{g}-\boldsymbol{h})}{\mathbf{2}\boldsymbol{g}+\mathbf{1}} - \mathbf{1}$$

∗ These numbers are called **local signatures**.

(defined by Endo '00, $\boldsymbol{g}=\mathbf{1},\mathbf{2}$ :Matsumoto ).

**Theorem B. (Sato-H)**

$c \subset \Sigma_g$ :non-separating s.c.c. $\iota_g(c) = c$

$^{\exists}h : \mathcal{H}_g(c) \to \mathbb{Q}$: homomorphism s.t.

$f : M \to S^2$ :HSBLF w/ genus-$g$, non-empty indef. fold.

whose vanishing cycle is $c$, then,

$$\sigma(M) = \sum_{i=1}^{n} \sigma_{\mathrm{loc}}(f^{-1}(y_i)) + h(t_{c_1} \cdot \cdots \cdot t_{c_n}),$$

where $f$(Lef. sing.) $= \{y_1, \ldots, y_n\}$

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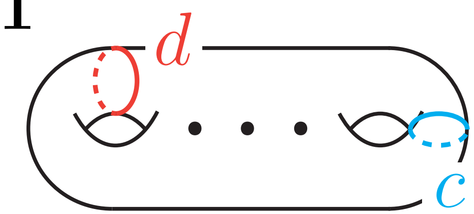
$*\ H_1(\mathcal{H}_g(c); \mathbb{Q}) = \mathbb{Q}[t_d]$ and $h([t_d]) = -\dfrac{1}{4g^2 - 1}$,



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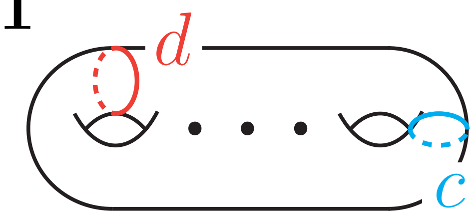
* $H_1(\mathcal{H}_g(c); \mathbb{Q}) = \mathbb{Q}[t_d]$ and $h([t_d]) = -\dfrac{1}{4g^2 - 1}$,

* $f$ is an LF $\Longrightarrow \sigma(M) = \sum_{i=1}^{n} \sigma_{\mathrm{loc}}(f^{-1}(y_i))$ (Endo, Matsumoto).



Thank you for your attention !