Quantum and homological representations of braid groups

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Plan

- Homological representations
  - Lawrence (1990's)
  - Bigelow, Krammer (2000) "Braid groups are linear"

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- Relation to KZ connection (comparison theorem)
- Quantum representations of mapping class groups (joint work with L. Funar)
  - Squier conjecture
  - Images of quantum representations
  - Index finite subgroups of mapping class groups

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$

$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

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Suppose $X = D$ (two dimensional disc).

$$\pi_1(\mathcal{F}_n(X)) = P_n, \quad \pi_1(\mathcal{C}_n(X)) = B_n$$

Relative configuration spaces

Fix $Q = \{(1,0), \cdots, (n,0)\} \subset D$. $\Sigma = D \setminus Q$

$$\mathcal{F}_{n,m}(D) = \mathcal{F}_m(\Sigma), \quad \mathcal{C}_{n,m}(D) = \mathcal{F}_m(\Sigma)/\mathfrak{S}_m$$

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Consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.

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Composing with the abelianization map
$\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}.$$

$\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$: the covering corresponding to $\mathrm{Ker}\,\beta$.

Homological representations

$H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module by deck transformations.

Express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$.

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

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$H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \begin{pmatrix} m+n-2 \\ m \end{pmatrix}.$$

$B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m}$: LKB representations $(m > 1)$

KZ connections

$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

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We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$: fiber of π, $\quad Y_{n,m} = X_{n,m}/\mathfrak{S}_m$

Representations of $sl_2(\mathbf{C})$

$\mathfrak{g} = sl_2(\mathbf{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda \in \mathbf{C}$

M_λ : Verma module of $sl_2(\mathbf{C})$ with highest weight vector v such that

$$Hv = \lambda v, \ Ev = 0$$

M_λ is spanned by

$$v, Fv, F^2 v, \cdots$$

Spece of null vectors

$\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n$
Consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$.
m : non-negative integer

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\}$$

The space of null vectors is defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}.$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.
The monodromy of KZ connection

$$\theta_{\kappa,\lambda} : B_n \longrightarrow \mathrm{Aut}\, N[|\Lambda| - 2m]$$

Comparison theorem

We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$.

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Theorem

There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the Lawrence-Krammer-Bigelow representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Outline of proof (1)

For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1,\cdots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1 < t_1 < \cdots < t_{m_1} < 2 \\
&2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\
&\cdots \\
&n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.
\end{aligned}$$

We put $M = (m_1, \cdots, m_{n-1})$ and write Δ_M for $\Delta_{m_1,\cdots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of Δ_M by the projection map $\pi_{n,m}$. The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$. Under genericiry conditions $[\overline{\Delta}_M]$ form a basis of $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$.

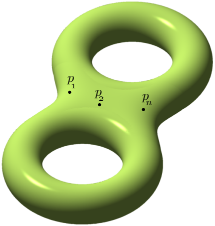
Outline of proof (2)

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is give by the matrix of the form

$$\left(\int_{\widetilde{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. Here $\omega_{M'}$ is a multivalued m-form on $X_{n,m}$. The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ is equivalent to the action of B_n on the solutions of the KZ equation with values in $N[n\lambda - 2m]$.

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights

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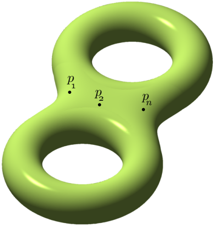

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$\mathcal{H}_\Sigma(p, \lambda)$: space of conformal blocks

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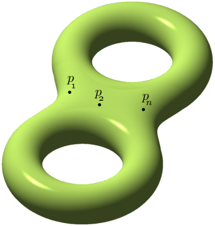


$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights
$\mathcal{H}_\Sigma(p, \lambda)$: space of conformal blocks
Geometry : vector bundle over the moduli space of Riemann surfaces with n marked points with projectively flat connection.

The mapping class group $\Gamma_{g,n}$ acts on $\mathcal{H}_\Sigma$:
Quantum representations

Gauss-Manin connection

$\mathcal{L}$: rank 1 local system over $Y_{n,m}$
$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})$
$\mathcal{H}_{n,m}$: local system over X_n with fiber $H_m(Y_{n,m}, \mathcal{L}^*)$

Theorem

There is surjective bundle map to the conformal block bundle

$$\mathcal{H}_{n,m} \longrightarrow \bigcup \mathcal{H}^*_{\mathbf{C}P^1}(p, \lambda)$$

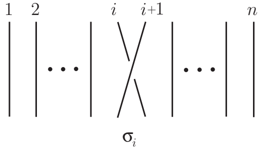
via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.

cf. Looijenga's work

Asymptotic faithfulness

Any two elements of the mapping class group are distinguished by the quantum representation for sufficiently large K (J. Andersen).

$B_n[k]$: normal subgroup of the braid group B_n generated by σ_i^k, $1 \leq i \leq n-1$.



Theorem (L. Funar and T. Kohno)

For any infinite set $\{k\}$, *we have* $\bigcap_k B_n[2k] = \{1\}$.

A positive answer to Squier's conjecture.

Images of quantum representations

The quantum representations are projectively unitary.

$$\rho_K : \Gamma_g \longrightarrow PU(\mathcal{H}_{\Sigma_g})$$

The k-th Johnson subgroup acts trivially on the k-th nilpotent quotients by the lower central series of the fundamental group $\pi_1(\Sigma_g)$.

The image of the quantum representation is "big" in the following sense.

Theorem (L. Funar and T. Kohno)

Suppose $g \geq 4$ and K sufficiently large. Then the image of any Johnson subgroup by ρ_K contains a non-abelian free group.