An algebraic formula of the WRT invariant for 3-manifolds

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Contents

Outline

Introduction

Known results

Invariants of 3-manifolds from $U_q(sl_2)$

- Quantum $SU(2)$ invariant (WRT invariant)
- Quantum $SO(3)$ invariant
- Hennings, Kauffman-Radford invariant (HKR invariant) from $U_q(sl_2)$

Each invariant can be interpreted as a Hennings invariant associated with an element z of the center of $U_q(sl_2)$.

Chen-Kuppum-Srinivasan 2008, Chen-Yu-Zhang 2010

The relations among these invariant were given.

e.x. (HKR invariant of M) $= |H_1(M)|$ (WRT invariant of M)

Integrality of invariants

Habiro

Integrality of WRT invariant for sl_2 and integral homology 3-spheres
(by a construction of unified WRT invariant for sl_2 and integral homology 3-sphere) $J_M(q) \in \hat{\mathbb{Z}}[q], J_M(q)|_{q=\zeta} = \tau_M(\zeta) \in \mathbb{Z}[\zeta]$

(Habiro-Le) for all simple Lie algebras and integral homology 3-spheres

Beliakova-Chen-Le

Integrality of quantum $SU(2)$ and $SO(3)$ invariant
(number theoretic proof)

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Integrality of HKR invariant Henning invariant with $z = 1$
(using property of the quantum double construction (Chen-Kerler))

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Problem

- Determine z which gives an invariant of 3-manifolds, and classify such invariants.
- Is all Hennings invariants associated with an element z of the center an algebraic integer?

Results

- We give an explicit presentation of z_{WRT} such that z_{WRT} gives the WRT invariant.
- We obtain another central element such that gives an invariant of 3-manifold.

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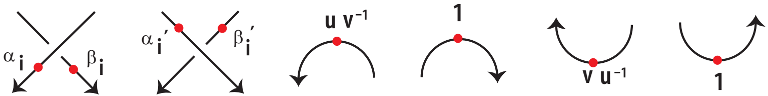
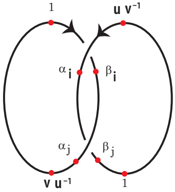
link invariant from a ribbon Hopf algebra

(A, R, v) : a ribbon Hopf algebra, $R = \sum \alpha_i \otimes \beta_i$, $R^{-1} = \sum \alpha'_i \otimes \beta'_i$

Invariant of an oriented link L

$L = L_1 \cup \cdots L_n \longrightarrow a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ $\quad u := \sum S(\beta_i)\alpha_i$





$\sum_{i,j} \beta_i \alpha_j v u^{-1} \otimes u v^{-1} \alpha_i \beta_j \in A \otimes A$

For $\mu_i \in (A/I)^*$, $\prod_{i=1}^{n} \mu_i(a_i)$ is an invariant of L. $\quad I = [A, A]$

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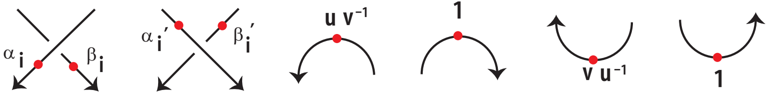
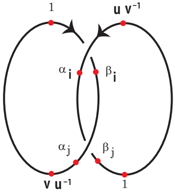




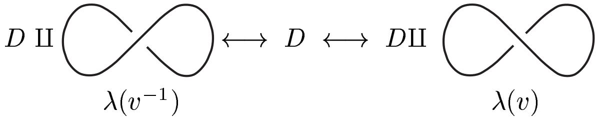
$$\sum_{i,j} \beta_i \alpha_j v u^{-1} \otimes u v^{-1} \alpha_i \beta_j \in A \otimes A$$

For $\mu_i \in (A/I)^*$, $\prod_{i=1}^n \mu_i(a_i)$ is an invariant of L. $\quad I = [A, A]$

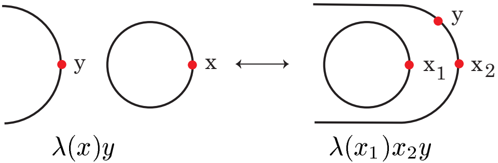
Definition

A : a finite dimensional Hopf algebra
$\lambda \in A^*$ is a right integral of A^*
$\overset{\text{def}}{\Leftrightarrow} m(\lambda \otimes 1)(\triangle(x)) = \lambda(x) \cdot 1$, *for* $\forall x \in A$

KI move



KII move



$\triangle(x) = \sum x_1 \otimes x_2$

Theorem (H,K-R)

(A, R, v) : *a 'unimodular' ribbon Hopf algebra,* $g = uv^{-1}$
$\lambda \in A^*$: *a right integral*

- $\mu(x) := \lambda(gx)$ *satisfies that*
  1) $\mu(xy) = \mu(yx)$, $x, y \in A$ $\quad \mu \in (A/I)^*$
  2) $\mu(S(x)) = \mu(x)$, $x, y \in A$
- $M = M_L$: *the result of Dehn surgery along a framed link* L *in* S^3
  If $\lambda(v), \lambda(v^{-1}) \neq 0$, *then*
  $$\phi(M) = \frac{\prod_{i=1}^{n} \mu(a_i)}{\mu([U_+])^{\sigma_+} \mu([U_-])^{\sigma_-}}$$ *is a topological invariant of* M.
  (HKR invariant)

  $U_\pm$: *the trivial knot with the framing* ± 1
  $\sigma_+(\sigma_-)$: *the number of positive (negative) eigenvalues of the linking matrix of* L

Note that $\mu([U_+]) = \lambda(v^{-1})$, $\mu([U_-]) = \lambda(v)$

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Proposition (H)

(A, R, v) : a unimodular finite dim. ribbon Hopf algebra,
λ : a right integral of A^*, $g = uv^{-1}$
For $\mu \in A^*$, (1) and (2) are equivalent.
(1) $\mu(xy) = \mu(yx)$, $\mu(x) = \mu(S(x))$
(2) $\exists z \in Z(A)$ s.t. $S(z) = z$, $\mu(x) = \lambda(gzx)$

Theorem (H)

If $z \in Z(A)$ satisfies that
$S(z) = z$, $(1 \otimes z)\triangle(z) = z \otimes z$, $\lambda(zv), \lambda(zv^{-1}) \neq 0$,
then $\lambda_z(x) := \lambda(gzx)$
$$\psi_{\lambda_z}(M) := \frac{(\lambda_z)^{\otimes n}([L])}{(\lambda_z([U_+]))^{\sigma_+}(\lambda_z([U_-]))^{\sigma_-}}$$ *is a top. invariant of M.*

Remark

$\phi(M) = \psi_{\lambda_1}(M)$ *(the case that $z = 1$ in the above theorem)*

Outline

$U_q(sl_2)$ case

$p \geq 2$, $q = \exp \frac{2\pi\sqrt{-1}}{2p}$

$U_q' = U_q(sl_2)$, $\mathbb{C}$-algebra gen. by E, F, k, k^{-1} with the relations

$$kE = qEk,\ kF = q^{-1}Fk,\ kk^{-1} = k^{-1}k = 1,\ EF - FE = \frac{k^2 - k^{-2}}{q - q^{-1}}$$

$$E^p = F^p = 0,\ k^{4p} = 1$$

Hopf algebra structure

$\triangle(k) = k \otimes k$, $\triangle(E) = 1 \otimes E + E \otimes k^2$, $\triangle(F) = 1 \otimes 1 + k^{-2} \otimes F$,
$S(k) = k^{-1}$, $S(E) = -Ek^2$, $S(F) = -k^2F$, $\varepsilon(E) = \varepsilon(F) = 0$, $\varepsilon(k) = 1$

Fact (U_q', R, v) is a unimodular ribbon Hopf algebra.

$\lambda \in {U_q'}^*$, $\lambda(F^m E^n k^j) = \delta_{m,p-1}\delta_{n,p-1}\delta_{j,2(p+1)}$: a right integral of U_q'

U_q : the Hopf subalgebra of U_q gen. by E, F, $K = k^2$, $K^{-1} = k^{-2}$

Proposition (Habiro)

For a framed n-component link L, $[L] \in U_q^{\otimes n}$.

Known results about $Z(U_q)$

- an explicit presentation of $Z(U_q)$ [FGST]
  $\mathbf{e}_i$ $(0 \le i \le p)$, $\mathbf{w}_j^{\pm}$ $(1 \le j \le p-1)$: a basis of $Z(U_q)$
  $\mathbf{e}_i\mathbf{e}_j = \delta_{ij}$, $\mathbf{e}_i\mathbf{w}_j^{\pm} = \delta_{ij}\mathbf{w}_i^{\pm}$, $\mathbf{w}_i^{\pm}\mathbf{w}_j^{\pm} = \mathbf{w}_i^{\pm}\mathbf{w}_j^{\mp} = 0$
  $\mathbf{C} := FE + \dfrac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$: Casimir element, $\beta_j := \dfrac{q^j + q^{-j}}{(q-q^{-1})^2}$
  $\pi_s^{+} := \frac{1}{2p}\sum_{n=0}^{s-1}\sum_{j=0}^{2p-1} q^{(2n-s+1)j}K^j$,
  $\pi_s^{-} := \frac{1}{2p}\sum_{n=s}^{p-1}\sum_{j=0}^{2p-1} q^{(2n-s+1)j}K^j$
  $\varphi_0(x) := (x-\beta_p)\prod_{r=1}^{p-1}(x-\beta_r)^2$, $\varphi_p(x) := (x-\beta_p)\prod_{r=1}^{p-1}(x-\beta_r)^2$
  $\varphi_s(x) := (x-\beta_0)(x-\beta_p)\prod_{\substack{r=1\\ r\neq s}}^{p-1}(x-\beta_r)^2$, $1 \le s \le p-1$,
  $\mathbf{w}_s^{\pm} := \pi_s^{\pm}\mathbf{w}_s$, $\mathbf{w}_s := \dfrac{1}{\varphi_s(\beta_s)}(\mathbf{C}-\beta_s)\varphi_s(\mathbf{C})(= \mathbf{w}_s^{+} + \mathbf{w}_s^{-})$
  $\mathbf{e}_s := \dfrac{1}{\varphi_s(\beta_s)}(\varphi_s(\mathbf{C}) - \varphi_s'(\beta_s)\mathbf{w}_s)\varphi_s(\mathbf{C})$, $\mathbf{w}_0, \mathbf{w}_p = 0$
- $\lambda(\mathbf{e}_j) = \lambda(\mathbf{w}_j) = 0$, $\lambda(\mathbf{w}_j^{+}) = (-1)^{j-1}\dfrac{[j]^3}{2p([p-1]!)^2}$

WRT invariant

$$tr_w(x) := \sum_{s=1}^{p-1}(-1)^{s-1}[s]\,\mathrm{tr}_{V_s}(x)$$

V_s : s-dim. irred. representation of U_q $\quad [s] := \dfrac{q^s - q^{-s}}{q - q^{-1}}$

- $\mathrm{tr}_w(K^{-p-1}\mathbf{w}_j^{\pm}) = 0$, $\mathrm{tr}_w(K^{-p-1}\mathbf{e}_j) = [j]^2$

Definition

$M = M_L$: the result of Dehn surgery along a framed link L in S^3

$\tau_p(M) := \dfrac{(\mathrm{tr}_w)^{\otimes n}([L])}{\mathrm{tr}_w([U_+])^{\sigma_+}\,\mathrm{tr}_w([U_-])^{\sigma_-}}$ *: WRT invariant of M*

Since $\mathrm{tr}_w(xy) = \mathrm{tr}_w(yx)$, $\mathrm{tr}_w(x) = \mathrm{tr}_w(S(x))$, by Proposition (H),

$$\exists z_{WRT} \in Z(U_q) \text{ s.t. } \mathrm{tr}_w(x) = \lambda(g z_{WRT} x), \ g = K^{p+1}$$

i.e. $\tau_p(M) = \psi_{\lambda_{z_{WRT}}}(M)$

Theorem (1)

$$z_{WRT} = \sum_{s=1}^{p-1} (-1)^{s-1} \frac{2p([p-1]!)}{[s]} \mathbf{w}_s^+$$

Theorem (2)

$z' := \frac{1}{2}(1 + K^p) \in Z(U_q)$

1. $S(z') = z'$, $(1 \otimes z')\triangle(z') = z' \otimes z'$
2. $z' = \sum_{s:odd} \mathbf{e}_s$
3. *If* $p \neq 0 \pmod 4$, *then* $\lambda(z'v), \lambda(z'v^{-1}) \neq 0$, *so we have a invariant* $\psi_{\lambda_{z'}}(M)$. *Moreover, if* p *is odd, then*

$$\psi_{\lambda_{z'}}(M) = |H_1(M)|\tau_p^{SO(3)}(M)$$

for a rational homology 3-sphere M.

Outline

A Proof of Theorem 1

Put $z_{WRT} := \sum_{s=0}^{p} a_s \mathbf{e}_s + \sum_{s=1}^{p-1} b_s \mathbf{w}_s + \sum_{s=1}^{p-1} c_s \mathbf{w}_s^+$.

Lemma

$a_s = 0\ (0 \leq s \leq p),\ c_s = (-1)^{s-1} \frac{2p([p-1]!)^2}{[s]}\ (1 \leq s \leq p-1)$

$\lambda(zx) = \sum_{j=1}^{p-1} (-1)^{j-1} [j]\, \mathrm{tr}_{V_j}(K^{-p-1} x)$

- $\lambda(z\mathbf{e}_s) = \sum_{j=1}^{p-1} (-1)^{j-1} [j]\, \mathrm{tr}_{V_j}(K^{-p-1} \mathbf{e}_s) = [s]^2$
  $(\mathrm{tr}_{V_j}(K^{-1}\mathbf{e}_s) = \delta_{js}[s])$
  $\lambda(z\mathbf{e}_s) = \lambda(a_s\mathbf{e}_s + b_s\mathbf{w}_s + c_s\mathbf{w}_s^+) = c_s\lambda(\mathbf{w}_s^+) \Rightarrow c_s = \frac{[s]^2}{\lambda(\mathbf{w}_s^+)}$
- $\lambda(z\mathbf{w}_s^+) = \lambda(a_s\mathbf{w}_s^+) = a_s\lambda(\mathbf{w}_s^+)$,
  $\lambda(z\mathbf{w}_s^+) = \sum_{j=1}^{p-1} (-1)^{j-1} [j]\, \mathrm{tr}_{V_j}(K^{-p-1} \mathbf{w}_s^+) = 0$,
  As $\lambda(\mathbf{w}_s^+) \neq 0$, $a_s = 0\ (1 \leq s \leq p-1)$.
- $\lambda(zF^{p-1}E^{p-1}K^p) \Rightarrow a_0 - a_p = 0$
  $\lambda(zF^{p-1}E^{p-1}K^{p+1}) \Rightarrow a_0 + a_p = 0$

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Lemma

$$\sum_{s=1}^{p-1} b'_s(-1)^{s\varepsilon}(f(s,l)+(-1)^l f(p-s,l)) = 0,\ \varepsilon = 0,1$$

for $0 \le l \le [\frac{p-1}{2}]$. $b'_s := \frac{b_s}{w_s}$, $w_s = \frac{p\sqrt{2p}}{[s]^2}$,

$f(s,l) := ([l]!)^2 \sum_{n=0}^{s-1} \begin{bmatrix} s-n+l-1 \\ l \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}$, $\begin{bmatrix} n \\ l \end{bmatrix} := \frac{[n]!}{[i]![n-i]!}$

the case $\varepsilon = 1$ $\quad \zeta := \sqrt{\frac{p}{2}}\frac{1}{([p-1]!)^2}$ $\quad z = \sum_{s=1}^{p-1} b_s \mathbf{w}_s + \sum_{s=1}^{p-1} c_s \mathbf{w}_s^+$

$$w_s \mathbf{w}_s^+ = \zeta \sum_{n=0}^{s-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1} ([i]!)^2 q^{j(s-1-2n)} \begin{bmatrix} s-n+i-1 \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} F^{p-1-i}E^{p-1-i}K^j$$

$w_s\lambda(F^l\mathbf{w}_s^+E^lK) = \zeta(-1)^{s-1}f(s,l)$, $\lambda(F^mE^nk^j) = \delta_{m,p-1}\delta_{n,p-1}\delta_{j,p+1}$

$w_s\lambda(F^l\mathbf{w}_s^-E^lK) = \zeta(-1)^l(-1)^{s-1}f(p-s,l)$

$\lambda(zF^lE^lK) =$

$\sum b_s(-1)^{s-1}\frac{\zeta}{w_s}(f(s,l)+(-1)^l f(p-s,l)) + \sum c_s\frac{\zeta}{w_s}(-1)^{s-1}f(s,l)$

$\lambda(zF^lE^lK) = \sum_{j=1}^{p-1}(-1)^{j-1}[j]\,\mathrm{tr}_{V_j}(K^pF^lE^l) = \sum_{j=1}^{p-1}[j]f(j,l)$

The case $\varepsilon = 0$, $K \to K^{p+1}$

Lemma

$$\sum_{s=1}^{p-1} b'_s(-1)^{s\varepsilon}(f(s,l)+(-1)^l f(p-s,l))=0,\ \varepsilon=0,1$$

for $0\le l\le[\frac{p-1}{2}]$. $b'_s:=\frac{b_s}{w_s}$, $w_s=\frac{p\sqrt{2p}}{[s]^2}$,

$f(s,l):=([l]!)^2\sum_{n=0}^{s-1}\begin{bmatrix}s-n+l-1\\ l\end{bmatrix}\begin{bmatrix}n\\ l\end{bmatrix}$, $\begin{bmatrix}n\\ l\end{bmatrix}:=\frac{[n]!}{[i]![n-i]!}$

the case $\varepsilon=1$ $\zeta:=\sqrt{\frac{p}{2}}\frac{1}{([p-1]!)^2}$ $z=\sum_{s=1}^{p-1}b_s\mathbf{w}_s+\sum_{s=1}^{p-1}c_s\mathbf{w}^+_s$

$w_s\mathbf{w}^+_s$

$$=\zeta\sum_{n=0}^{s-1}\sum_{i=0}^{n}\sum_{j=0}^{2p-1}([i]!)^2q^{j(s-1-2n)}\begin{bmatrix}s-n+i-1\\ i\end{bmatrix}\begin{bmatrix}n\\ i\end{bmatrix}F^{p-1-i}E^{p-1-i}K^j$$

$w_s\lambda(F^l\mathbf{w}^+_sE^lK)=\zeta(-1)^{s-1}f(s,l)$, $\lambda(F^mE^nk^j)=\delta_{m,p-1}\delta_{n,p-1}\delta_{j,p+1}$

$w_s\lambda(F^l\mathbf{w}^-_sE^lK)=\zeta(-1)^l(-1)^{s-1}f(p-s,l)$

$\lambda(zF^lE^lK)=$

$\sum b_s(-1)^{s-1}\frac{\zeta}{w_s}(f(s,l)+(-1)^lf(p-s,l))+\sum c_s\frac{\zeta}{w_s}(-1)^{s-1}f(s,l)$

$\lambda(zF^lE^lK)=\sum_{j=1}^{p-1}(-1)^{j-1}[j]\,\mathrm{tr}_{V_j}(K^pF^lE^l)=\sum_{j=1}^{p-1}[j]f(j,l)$

The case $\varepsilon=0$, $K\to K^{p+1}$

Lemma

$b'_s = 0$, *i.e.* $b_s = 0$ $(1 \le s \le p-1)$

$$\sum_{s=1}^{p-1} b'_s(-1)^{s\varepsilon}(f(s,l) + (-1)^l f(p-s,l)) = 0, \varepsilon = 0, 1$$

$A(s) := q^s + q^{-s}$, $[[ls]] := \frac{[ls]}{[l]}$, $B_i^{(j)} \in \mathbb{Z}[q]$, $\{n\} := q^n - q^{-n}$

$$f(s,l) = \sum_{n=0}^{s-1} \prod_{k=0}^{l-1} (\{s-n+k\}\{n-k\})$$
$$= \sum_{n=0}^{s-1} \prod_{k=0}^{l-1} \left(A(s) - (q^{s+2k}q^{-2n} + q^{-s-2k}q^{2n})\right)$$
$$= sA(s)^l - A(s)^{l-1}[[s]]B_{l-1}^{(1)}(q) + A(s)^{l-2}\{[[2s]]B_{l-2}^{(2)}(q) + sB_{l-2}^{(0)}(q)\}$$
$$\ldots \begin{cases} -\sum_{k=0}^{\frac{l-1}{2}}[[(l-2k)s]]B_0^{(l-2k)}(q) & l : \text{odd} \\ +\sum_{k=0}^{\frac{l-2}{2}}[[(l-2k)s]]B_0^{(l-2k)}(q) + sB_0^{(0)}(q) & l : \text{even} \end{cases}$$

Note that $A(p-s) = -A(s)$ and $[[l(p-s)]] = (-1)^{l-1}[[ls]]$.

$$f(s,l) + (-1)^l f(p-s,l) = p\sum_{k=0}^{[\frac{l}{2}]} A(s)^{l-2k}B_{l-2k}^{(0)}(q)$$

Noting that $A(s) \ne A(s')$ if $|s-s'| \le p-1$, we obtain $b'_s = 0$.

Outline

Proof of Theorem 2

$z' = \frac{1}{2}(1 + K^p)$

(1) From $S(K^p) = K^{-p} = K^p$, $S(z') = z'$.
$(1 \otimes z')\triangle(z') = \frac{1}{4}(1 \otimes (1 + K^p))(1 \otimes 1 + K^p \otimes K^p) = z' \otimes z'$

(2) From the actions of z' to the irreducible representations and the projective representations of U_q [FGST], we obtain $z' = \sum_{s:odd} \mathbf{e}_s$.

(3)

(FGST)
$v^{\pm} = \sum_{s=0}^{p}(a_{\pm,s}\mathbf{e}_s + b_{\pm,s}\mathbf{w}_s + c_{\pm,s}\mathbf{w}_s^{+})\ (w_0^{\pm} = w_p^{\pm} = 0)$
$a_{\pm,s} = (-1)^{s-1}q^{\pm\frac{1-s^2}{2}}$, $b_{\pm,s} = \pm(-1)^{p-1}(q - q^{-1})q^{\pm\frac{1-s^2}{2}}\frac{s}{[s]}$,
$c_{\pm,s} = -\frac{pb_{\pm,s}}{s}$

$$\lambda(z'v^{\pm}) = \sum_{s:odd}^{p-1} c_{\pm,s}\lambda(\mathbf{w}_s^{+}) = \pm\frac{q - q^{-1}}{2([p-1]!)^2}\sum_{s:odd}^{p-1} q^{\pm\frac{1-s^2}{2}}[s]^2$$
$$\neq 0 \text{ if } p \neq 0 \pmod 4)$$

Action of an element of the center of U_q on irred. and proj. repr.s

$z = \sum_{s=0}^{p} a_s \mathbf{e}_s + \sum_{s=1}^{p-1}(c_s^+ \mathbf{w}_s^+ + c_s^- \mathbf{w}_s^-)$

- $z d_n^s = a_s d_n^s$ in $\mathcal{X}^+(s)$: s-dim irred. repr. $(1 \leq s \leq p)$
  $K d_n^s = q^{s-1-2n} d_n^s$
  $(1 + K^p) d_0^s = (1 + q^{p(s-1)}) d_0^s = (1 + (-1)^{s-1}) d_0^s$
- $z b_n^{(+,s)} = c_s^+ a_n^{(+,s)}$ in $\mathcal{P}^+(s)$: proj. repr. $(1 \leq s \leq p-1)$
  $K b_n^{(+,s)} = q^{s-1-2n} b_n^{(+,s)}$
- $z y_n^{(-,s)} = c_s^- x_n^{(-,s)}$ in $\mathcal{P}^-(s)$: proj. repr. $(1 \leq s \leq p-1)$
  $K y_n^{(-,s)} = -q^{p-s-1-2n} y_n^{(-,s)}$

$\mathbf{w}_s^+ b_n^{(+,s)} = a_n^{(+,s)}$, $\mathbf{w}_s^- y_n^{(-,s)} = x_n^{(-,s)}$

Action of an element of the center of U_q on irred. and proj. repr.s

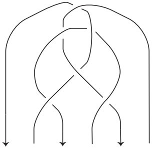
$z = \sum_{s=0}^{p} a_s \mathbf{e}_s + \sum_{s=1}^{p-1}(c_s^+ \mathbf{w}_s^+ + c_s^- \mathbf{w}_s^-)$

- $zd_n^s = a_s d_n^s$ in $\mathcal{X}^+(s)$: s-dim irred. repr. $(1 \le s \le p)$
  $Kd_n^s = q^{s-1-2n} d_n^s$
  $(1 + K^p)d_0^s = (1 + q^{p(s-1)})d_0^s = (1 + (-1)^{s-1})d_0^s$
- $zb_n^{(+,s)} = c_s^+ a_n^{(+,s)}$ in $\mathcal{P}^+(s)$: proj. repr. $(1 \le s \le p-1)$
  $Kb_n^{(+,s)} = q^{s-1-2n} b_n^{(+,s)}$
- $zy_n^{(-,s)} = c_s^- x_n^{(-,s)}$ in $\mathcal{P}^-(s)$: proj. repr. $(1 \le s \le p-1)$
  $Ky_n^{(-,s)} = -q^{p-s-1-2n} y_n^{(-,s)}$

$\mathbf{w}_s^+ b_n^{(+,s)} = a_n^{(+,s)}$, $\mathbf{w}_s^- y_n^{(-,s)} = x_n^{(-,s)}$

$\psi_{\lambda_{z'}}(M) = |H_1(M)|\tau_p^{SO(3)}(M)$

L : a framed links s.t it is a closure of a bottom tangle T.



bottom tangle

M : a closed oriented 3-manifold obtained by Dehn surgery along L

$$\tau_p^{SO(3)}(M) = \frac{(\mathrm{tr}_q^{w'})^{\otimes n}([T])}{\mathrm{tr}_q^{w'}(v^{-1})^{\sigma_+}\,\mathrm{tr}_q^{w'}(v)^{\sigma_-}},\ \mathrm{tr}_q^{w'}(x) := \sum_{s:odd}^{p-1}[s]\,\mathrm{tr}_{V_s}(Kx)$$

$$\psi_{\lambda_{z'}}(M) = \frac{(\lambda^{z'})^{\otimes n}([T])}{\lambda^{z'}(v^{-1})^{\sigma_+}\lambda^{z'}(v)^{\sigma_-}},\ \lambda^{z'}(x) = \lambda(z'x)$$

For a rational homology 3-sphere M, $\exists L(n_i, 1)$ $(1 \leq i \leq m)$ s.t. $M' = M\sharp_{i=1}^m L(n_i, 1)$ is obtained by Dehn surgery on a link with diagonal linking matrix. (Ohtsuki)

$\psi_{\lambda_{z'}}(M_1 \sharp M_2) = \psi_{\lambda_{z'}}(M_1)\psi_{\lambda_{z'}}(M_2)$,
$\tau_p^{SO(3)}(M_1 \sharp M_2) = \tau_p^{SO(3)}(M_1)\tau_p^{SO(3)}(M_2)$,
$\psi_{\lambda_{z'}}(L(m,1)) = m\tau_p^{SO(3)}((L(m,1)) \neq 0$

$$\bar{U}_q := U_q/\{a(x) - \varepsilon(a)x, a, x \in U_q\}, a(x) = \sum a_1 x S(a_2)$$
$$\triangle(a) = \sum a_1 \otimes a_2$$

Lemma (Habiro)

If $z \in Z(U_q)$*, then* λ^z, $(\mathrm{tr}_q^w)^z$ *factor through* $\bar{U}_q$*, where*

$$\lambda^z(x) := \lambda(xz), \ (\mathrm{tr}_q^w)^z(x) := \mathrm{tr}_q^w(xz), \ \forall \in U_q$$

Lemma (CYZ)

T *: m-comp. bottom tangle s.t.* $\hat{T}$ *has* 0 *linking matrix.*
If $\chi_i : U_q \to \mathbb{C}$ *factors through* $\bar{U}_q$*, then*

$$(\mathrm{Id} \otimes \chi_2 \otimes \cdots \chi_m)([T]) \in \bar{Z}(U_q),$$

$\bar{Z}(U_q)$ *: the subset of* $Z(U_q)$ *spanned by*
$\mathbf{e}_s$ $(0 \le s \le p)$, $\mathbf{w}_s$ $(1 \le s \le p-1)$

Lemma

If $x \in \bar{Z}(U_q)$, $\dfrac{\lambda^{z'}(xv^{\pm n})}{\lambda^{z'}(v^{\pm})} = n\dfrac{\mathrm{tr}_q^{w'}(xv^{\pm n})}{\mathrm{tr}_q^{w'}(v^{\pm})}$, $\forall n \in \mathbb{N}$

$$v^{\pm n} = \sum_{s=0}^{p}(a_{\pm,s}\mathbf{e}_s + na_{\pm,s}^{n-1}(b_{\pm,s}\mathbf{w}_s + c_{\pm,s}\mathbf{w}_s^{+}))$$

For $x = \mathbf{e}_j$ with odd j,

$$\begin{aligned}
\frac{\lambda^{z'}(xv^{\pm n})}{\lambda^{z'}(v^{\pm n})} &= \frac{\lambda(z'\mathbf{e}_j v^{\pm n})}{\lambda(z'v^{\pm n})} = n\frac{a_{\pm,j}^{n-1}c_{\pm,j}\lambda(\mathbf{w}_j^{+})}{\sum_{s:odd} c_{\pm,s}\lambda(\mathbf{w}_s^{+})} \\
&= n\frac{(-1)^{(j-1)n}q^{\pm\frac{(1-j^2)n}{2}}[j]^2}{\sum_{s:odd}(-1)^s q^{\pm\frac{(1-s^2)}{2}}[s]^2} \\
&= n\frac{a_{\pm,j}^{n}\,\mathrm{tr}_q^{w'}(\mathbf{e}_j)}{\sum_{s:odd} a_{\pm,s}\,\mathrm{tr}_q^{w'}(\mathbf{e}_s)} = n\frac{\mathrm{tr}_q^{w'}(xv^{\pm n})}{\mathrm{tr}_q^{w'}(v^{\pm n})}
\end{aligned}$$

For $x = \mathbf{e}_j$ with even j and $x = \mathbf{w}_j$, $LHS = RHS = 0$.

T : n-comp. bottom tangle
s.t. $\hat{T}$ has diagonal linking matrix $(f_1, f_2, \dots, f_n)$
M : the result of surgery on $\hat{T}$, $|H_1(M)| = |f_1 \cdots f_n|$
suppose $f_1, \dots, f_i > 0$, $f_{i+1}, \dots, f_n < 0$

$$\begin{aligned}
\psi_{\lambda_{z'}}(M) &= \frac{(\lambda^{z'})^{\otimes n}([T])}{\lambda^{z'}(v^{-1})^{\sigma_+}\lambda^{z'}(v)^{\sigma_-}} = \frac{\lambda^{z' v^{-f_1}} \otimes \cdots \otimes \lambda^{z' v^{f_n}}([T_0])}{\lambda(z'v^{-1})^i \lambda(z'v)^{n-i}} \\
&= \frac{\lambda^{z' v^{-f_1}}}{\lambda(z'v^{-1})} \left(\frac{\mathrm{Id} \otimes \lambda^{z' v^{-f_2}} \otimes \cdots \otimes \lambda^{z' v^{f_n}}([T_0])}{\lambda(z'v^{-1})^{i-1}\lambda(z'v)^{n-i}} \right) \in \bar{Z}(U_q) \\
&= |f_1| \frac{(\mathrm{tr}_q^{w'})^{v^{-f_1}}}{\mathrm{tr}_q^{w'}(v^{-1})} \left(\frac{\mathrm{Id} \otimes \lambda^{z' v^{-f_2}} \otimes \cdots \otimes \lambda^{z' v^{f_n}}([T_0])}{\lambda(z'v^{-1})^{i-1}\lambda(z'v)^{n-i}} \right) \\
&= \cdots \\
&= |f_1 \cdots f_n| \frac{(\mathrm{tr}_q^{w'})^{v^{-f_1}} \otimes (\mathrm{tr}_q^{w'})^{v^{f_n}}([T_0])}{\mathrm{tr}_q^{w'}(v^{-1})^i \, \mathrm{tr}_q^{w'}(v)^{n-i}} = |H_1(M)| \tau_p^{SO(3)}(M)
\end{aligned}$$

T : n-comp. bottom tangle
s.t. $\hat{T}$ has diagonal linking matrix $(f_1, f_2, \dots, f_n)$
M : the result of surgery on $\hat{T}$, $|H_1(M)| = |f_1 \cdots f_n|$
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$$
\begin{aligned}
\psi_{\lambda_{z'}}(M) &= \frac{(\lambda^{z'})^{\otimes n}([T])}{\lambda^{z'}(v^{-1})^{\sigma_+}\lambda^{z'}(v)^{\sigma_-}} = \frac{\lambda^{z'v^{-f_1}} \otimes \cdots \otimes \lambda^{z'v^{f_n}}([T_0])}{\lambda(z'v^{-1})^i \lambda(z'v)^{n-i}} \\
&= \frac{\lambda^{z'v^{-f_1}}}{\lambda(z'v^{-1})} \left(\frac{\mathrm{Id} \otimes \lambda^{z'v^{-f_2}} \otimes \cdots \otimes \lambda^{z'v^{f_n}}([T_0])}{\lambda(z'v^{-1})^{i-1}\lambda(z'v)^{n-i}} \right) \in \bar{Z}(U_q) \\
&= |f_1| \frac{(\mathrm{tr}_q^{w'})^{v^{-f_1}}}{\mathrm{tr}_q^{w'}(v^{-1})} \left(\frac{\mathrm{Id} \otimes \lambda^{z'v^{-f_2}} \otimes \cdots \otimes \lambda^{z'v^{f_n}}([T_0])}{\lambda(z'v^{-1})^{i-1}\lambda(z'v)^{n-i}} \right) \\
&= \cdots \\
&= |f_1 \cdots f_n| \frac{(\mathrm{tr}_q^{w'})^{v^{-f_1}} \otimes (\mathrm{tr}_q^{w'})^{v^{f_n}}([T_0])}{\mathrm{tr}_q^{w'}(v^{-1})^i \, \mathrm{tr}_q^{w'}(v)^{n-i}} = |H_1(M)| \tau_p^{SO(3)}(M)
\end{aligned}
$$

Outline

Conclusion

$$\phi(M) = \psi_{\lambda_1}(M) = |H_1(M)|\tau_p(M),\ 1 = \sum_{s=0}^{p} \mathbf{e}_s$$

$$\tau_p(M) = \psi_{\lambda_{z_{WRT}}}(M),\ z_{WRT} = \sum_{s=1}^{p-1}(-1)^{s-1}\frac{2p([p-1]!)}{[s]}\mathbf{w}_s^+$$

$$\psi_{\lambda_{z'}}(M) = |H_1(M)|\tau_p^{SO(3)}(M),\ z' = \sum_{s:odd} \mathbf{e}_s$$

$$\tau_p^{SO(3)}(M) = \psi_{\lambda_{z'_{WRT}}}(M),\ z'_{WRT} = \sum_{s:odd}\frac{2p([p-1]!)}{[s]}\mathbf{w}_s^+ \ (\ p : \text{odd})$$

Remark

For $p = 3$, $(1 \otimes z_{WRT})\triangle(z_{WRT}) \neq z_{WRT} \otimes z_{WRT}$.

$$\{z \in Z(U_q) \mid (1 \otimes z)\triangle(z) = (z \otimes z),\ S(z) = z,\ \lambda(zv^{\pm}) \neq 0\}$$
$$\neq \{z \in Z(U_q) \mid \psi_{\lambda_z} \text{ is an invariant of } M\}$$

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