Some new results on new knot invariants

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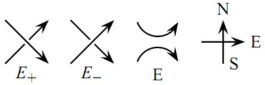
Jan 9-12, Kaist

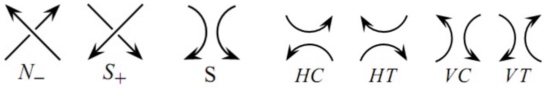
Outline of the talk

- A knot invariant with two skein relations.
- Detecting mutations
- The linear regional invariant.
- General regional invariant.

The construction of a knot invariant



The Possible smoothing methods.

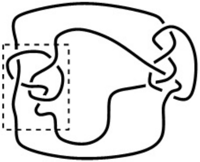
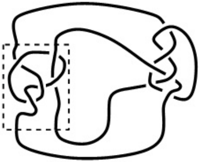


We use the following skein relations to define a regular isotopy knot invariant $f(D)$.
If the two arcs are from the same component,
$f(E_+) + bf(E_-) + df(VT) = 0$,
otherwise, $f(E_+) + b'f(E_-) + cf(E) = 0$.
If D a trivial n-component link 0-crossing diagram, then $f(D) = v_n$.

(1) If the two arcs are from the same component,
$f(E_+) + bf(E_-) + df(VT) = 0$,
otherwise, $f(E_+) + b'f(E_-) + cf(E) = 0$.
(2) For a n-component link trivial diagram D, $f(D) = v_n$.
(3) $b'c = bc, cd = 0, dd = 0, b^2d = d$.
$\{h(w) + h(w-2)b + dh(w-1)\}v_n = 0$.
Here w is the writhe, $h : N \to R$ with value in some commutative ring R, each $h(w)$ is invertible in R. For example, $h(w) = a^w$.

Then $F(D) = h(w(D))f(D)$ is a knot invariant.

Why certain invariants does not detect Mutations

$11^n_{34} = \overline{C} =$  $11^n_{42} = \overline{KT} =$ 

- **(a) Crossings: 1-1 correspondence.**

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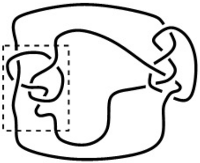
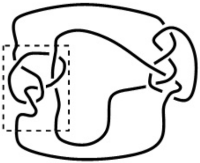
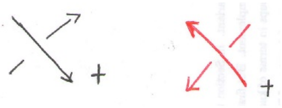
$11^n_{34} = \overline{C} =$  $11^n_{42} = \overline{KT} =$ 

- **(a) Crossings: 1-1 correspondence.**
- A Mutation operation either changes all the orientation of the two strings, or changes non.



- **(b) Writhe: same.**

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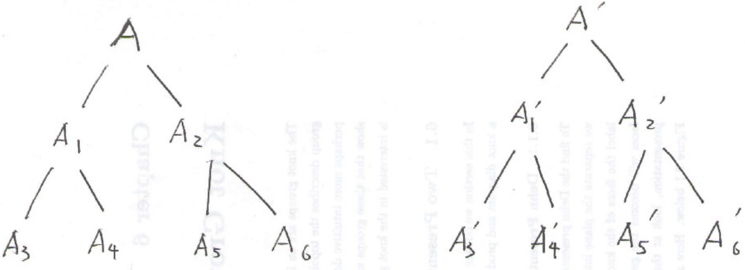

- **There is a canonical 1-1 correspondence between their resolution/calculations (same skein tree).**
- **Each A_i, A_i' are mutations of each other, same writhe.**

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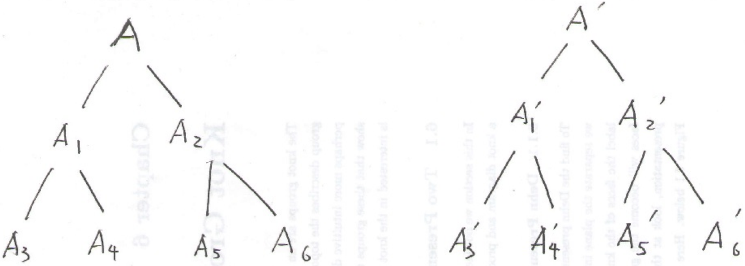

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- For example, the HOMFLY, Kauffman 2-variable, colored Jones polynomial, the hyperbolic volume.

Why this invariant may detect Mutations

If the two arcs are from the same component,
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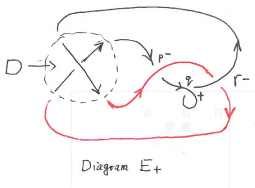

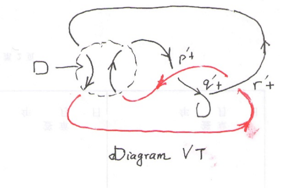

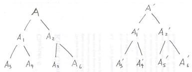

- The orientation change is global.
- The crossings changes $(p-, q+, r-) \Rightarrow (p'+, q'+, r'+)$

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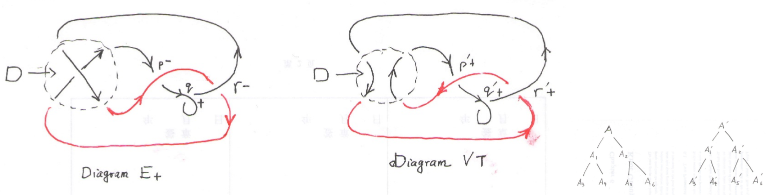

- The orientation change is global.
- The crossings changes $(p-, q+, r-) \Rightarrow (p'+, q'+, r'+)$
- The mutations do not have the same skein tree, writhe in the resolution diagrams.

Calculation

Most knot invariants do not distinguish Mutations. For example, the HOMFLY, Kauffman 2-variable, colored Jones polynomial, the hyperbolic volume.

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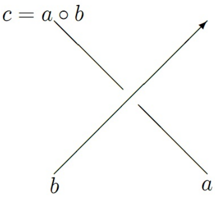
- The famous first mutants pair in the knot table are the Conway knot C (11_{n34}) of genus 3 and the Kinoshita-Terasaka (11_{n42}) knot KT of genus 2.
- $F(KT) = h(-1)\{b^{-2}h^{-1}(3) + dh^{-1}(2) - b^{-1}d\}v_1$
  $F(C) = h(-1)\{bd + h^{-1}(-1) - dbh^{-1}(4)\}v_1$.

Algebra /Gröbner Basis

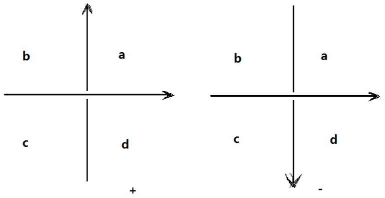
There is a slightly weaker invariant can also distinguish mutants. This time we ask $h(n) = a^n$ for some invertible variable a. We can now ask $b = b', B = B', c = d$, then we have $a^2 + b + da = 0$. Let $A > a > B > b > d$, then a Gröbner basis is $b^2d - d, dd, bB - 1, Bd - bd, aA - 1, a^2 + b + da, bA + a + d, A + aB + bd, dA + abd, bdA + ad$.
Now $a^3b^2F(KT) = a + da^2 - bda^4 = a - 2bd$, $a^3b^2F(C) = bda^4 - bd + a^3b^2 = bd - ab^3$. Hence this invariant distinguishes mutants.

Regional invariants

- 

The local quandle relation.
- Color each arc of the link diagram.

An example

- 

- $ab^{-1}cd^{-1} = 1.$
- $S = < a, b, c, d, \cdots \mid equations >$

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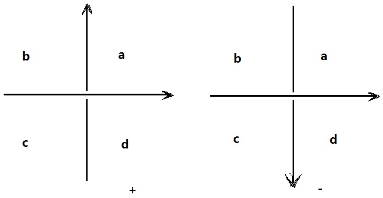
- 

- $ab^{-1}cd^{-1} = 1.$
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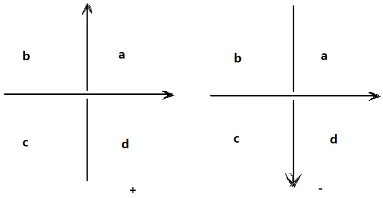
- This gives the "Dehn presentation of the knot group" free product with Z.

A linear Regional invariant: color the regions.

- 

- We introduce the following equations: $a + xb + zc + yd = 0$ and $b + xa + zd + yc = 0$.
  Here x, y, z are invertible parameters.

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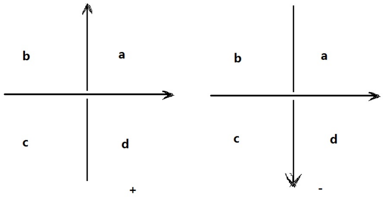
- 

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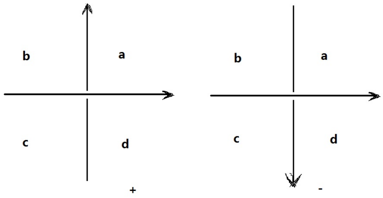
- 

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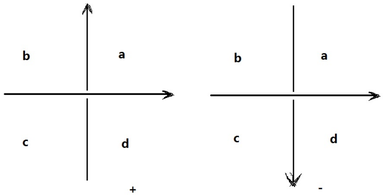
- 

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  Here x, y, z are invertible parameters.
- The two equations guarantee the Redeimeister I, II invariance, another condition $z = xy$ guarantee the Redeimeister III invariance.
- If we let $x = 1, z = y$, this is equivalent to Alexander's equations to define the Alexander polynomial.

A linear Regional invariant

Then for any oriented link diagram D, we marked the regions by different symbols $a, b, c, d, \cdots$. At each crossing, we get a linear equation.
Then we get a $S =<< a, b, c, d, \cdots \mid equations >>$.
The set S is defined as follows.
In the ring $R[x, x^{-1}, y, y^{-1}](a, b, c, d, \cdots)$ inductively define $s_0 = \{a, b, c, d, \cdots\}$
If any three of $\alpha, \beta, \gamma, \delta \in s_n$, then the equation $\alpha + x\beta + z\gamma + y\delta = 0$ defines another element $s \in R[x, x^{-1}, y, y^{-1}](a, b, c, d, \cdots)$. We ask s_{n+1} is the minimal set to contain all such s and all element in s_n.
So $s_n \subset s_{n+1}$, and $\overline{S} = \cup s_i$.
Let S be $\overline{S}$ module the equations in the definition $S =<< a, b, c, d, \cdots \mid equations >>$.

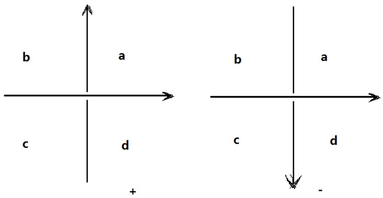
A linear Regional invariant

There are a few byproducts here.

1. Finite coloring. If we ask $x, y, a, b, c, d, \cdots \in Z_p$ for some finite field, the we get a coloring of the regions.
2. 2-variable ideal.
3. 2-variable polynomial. When $x = 1$, this is equivalent to the Alexander polynomial.

Question: Does the 2-variable polynomial always degenerate to the Alexander polynomial in z?

The general Regional invariant

- 

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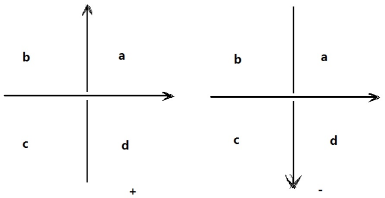
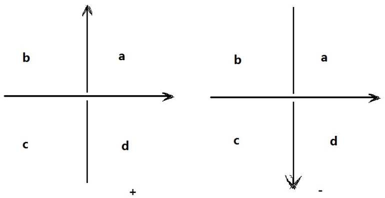
- 

- We introduce the following equations: $F(a, b, c, d) = 0$ and $G(a, b, c, d) = 0$.
  We require that in any of those equations, if we know three variables, then the forth one is uniquely determines. We also ask $F(a, b, c, d) = G(b, a, d, c)$ for any a, b, c, d. Those two conditions guarantee the Redeimeister I, II invariance. For Redeimeister III invariance, we need another 16 equations for the function F.

The general Regional invariant

- 

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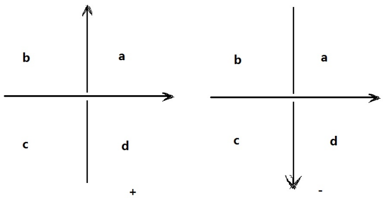
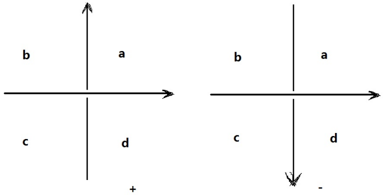
- 

- We introduce the following equations: $F(a, b, c, d) = 0$ and $G(a, b, c, d) = 0$.
  Like the linear case, we can also inductively build an invariant set $\overline{S} = \cup s_i$. This is a regional analogue of the knot quandle.

A few examples

- 

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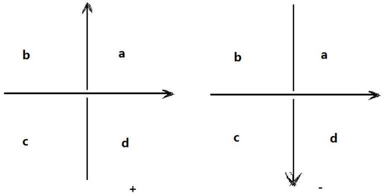
- 

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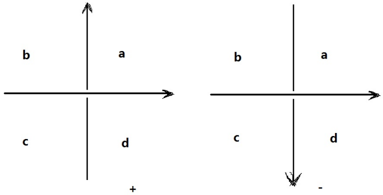
- 

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- 2. $F(a,b,c,d) = ab^{-1}cd^{-1}x$. Where x commutates with all variables.
- 3. $F(a,b,c,d) = a - b + c - d + \delta$. Or, equivalently, $F(a,b,c,d) = ab^{-1}cd^{-1}\delta$ where all variables commutate with each other.

Last page

Thanks!