Fixed subgroups of automorphisms of hyperbolic 3-manifold groups

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Abstract

For fixed subgroups $Fix(\phi)$ of automorphisms ϕ on hyperbolic 3-manifold groups $\pi_1(M)$, we observed that $\mathrm{rk}(Fix(\phi)) < 2\mathrm{rk}(\pi_1(M))$ and the constant 2 in the inequality is sharp, we also classify all possible groups $Fix(\phi)$.

1 Background and Main Theorem

For a group G and an automorphism $\phi : G \to G$, we define $Fix(\phi) = \{\omega \in G | \phi(\omega) = \omega\}$, and use $\mathrm{rk}(G)$ to denote the rank of G.

The so called Scott conjecture proved 20 years ago in a celebrate work of M. Bestvina and M. Handel [BH] states that:

Theorem 1.1. *For each automorphism ϕ on a free group $G = F_n$,*

$$rk(Fix(\phi)) \leq rk(G).$$

In a recent paper by B.J. Jiang, S. D. Wang and Q. Zhang [JWZ], it is proved that

Theorem 1.2. *For each automorphisms ϕ on a compact surface group $G = \pi_1(S)$,*

$$rk(Fix(\phi)) \leq rk(G).$$

It is obvious that the bound given in Theorem 1.1 and Theorem 1.2 are sharp.

It is natural to wonder to what degree such an inequality still hold for groups with good geometric background.

A main observation in this paper is the following

Theorem 1.3. *For each automorphism ϕ on a hyperbolic 3-manifold group $G = \pi_1(M)$,*

$$rk(Fix(\phi)) < 2rk(G),$$

and the upper bound is sharp when G runs over all hyperbolic 3-manifold groups.

Theorem 1.3 is a conclusion of the following Theorems 1.4, and 1.5.

Theorem 1.4. *There exist a sequences automorphisms $\phi_n : \pi_1(M_n) \to \pi_1(M_n)$ on closed hyperbolic 3-manifolds M_n such that $Fix(\phi_n)$ is the group of a closed surface, and*

$$\frac{rk(Fix(\phi_n))}{rk(\pi_1(M_n))} > 2 - \epsilon \text{ as } n \to \infty$$

for any $\epsilon > 0$.

Theorem 1.5. *Suppose ϕ is an automorphism on $G = \pi_1(M)$, where M is a hyperbolic 3-manifold. Then $rk(Fix(\phi)) < 2rk(G)$.*

To prove Theorem 1.5, we need the following Theorem 1.6

Theorem 1.6. *Suppose $G = \pi_1(M)$, where M is a hyperbolic 3-manifold, and ϕ is a automorphism of G. Then $Fix(\phi)$ is one of the following types:*

the whole group G; the trivial group $\{e\}$; $\mathbb{Z}$; $\mathbb{Z} \bigoplus \mathbb{Z}$; the surfaces group $\pi_1(S)$, where S can be orientable or not, and closed or not. More precisely

(1) Suppose ϕ is induced by an orientation preserving isometry.

(i) $Fix(\phi)$ is either $\mathbb{Z}$, or $\mathbb{Z} \bigoplus \mathbb{Z}$, or G, or $\{e\}$; moreover

(ii) if M is closed, then $Fix(\phi)$ is either $\mathbb{Z}$ or G;

(2) Suppose ϕ is induced by an orientation reversing isometry f.

(i) If $\phi^2 \neq id$, then $Fix(\phi)$ is either $\mathbb{Z}$ or $\{e\}$;

(ii) if $\phi^2 = id$, then $Fix(\phi)$ is either $\{e\}$, or the surface group $\pi_1(S)$, which is pointwisely fixed by f.

2 Construction of Examples 1.4

Roughly speaking those examples are constructed as follow: we first construct a hyperbolic 3-manifold P with totally geodesic boundary. Then we double it to get a closed hyperbolic 3 manifold DP. Now if we choose the base point on the boundary of P, the reflection along ∂P will induce ϕ on the fundamental group of DP, and this automorphism ϕ will have the property we desired.

In Thurston's Lecture Notes (Section 3.2 of [Th1]), there is a very concrete and beautiful construction of hyperbolic 3-manifolds with totally geodesic boundaries involving primary hyperbolic geometry only.

In 3-dimensional hyperbolic space H^3, there is a one-parameter family of truncated hyperbolic tetrahedron as in Figure 1: Each of its 8 faces is totally geodesic; each of its 18 edges is geodesic line segment. There are 4 triangle faces and 4 hexagon faces. The 12 edges of the 4 triangle faces have the same length, and the remain 6 edges, we call them "inner edge", also have the same length. The triangle faces are perpendicular to the hexagon faces. The angles between hexagon faces are all equal and can be arbitrary angles between $(0°, 60°)$.

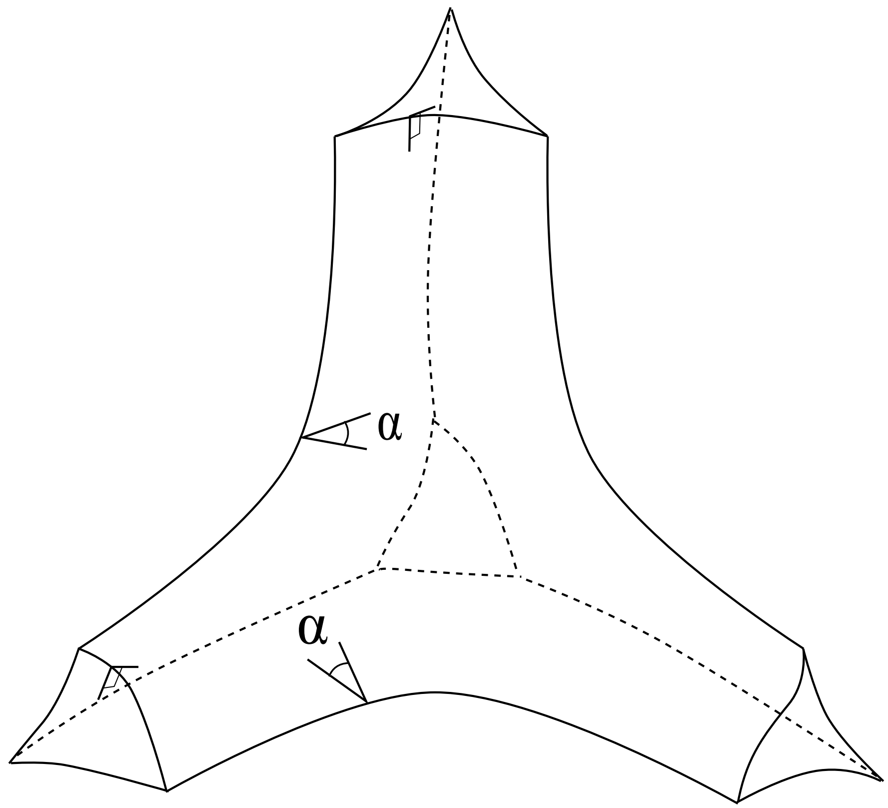

Figure 1

Suppose we have some copies of tetrahedron. We pair the faces of tetrahedron and gluing them together.After gluing, if we remove a neighborhood of the vertex, we will get a topological manifold P. A tetrahedron with its vertex neighborhood removed is homeomorphic to the truncated simplex mention above. Suppose every k edges of the tetrahedron are glued together($k > 6$). We can set the face angle α of the truncated simplex to be $\frac{2\pi}{k}$. Then the hyperbolic structure of the truncated simplex fix together to give the hyperbolic structure of P, and the triangle faces of the truncated simplex are matched together to form the totally geodesic ∂P.

It is easy to see that the number of vertex of tetrahedron (after gluing) equals the number of the boundary component.

Moreover, if we remove the neighborhood of the inner edges in P. We will get a handlebody H. To see this, we remove the neighborhood of the 6 edges of a tetrahedron. Topologically, it is homeomorphic to D^3 and the 4 tetrahedron faces are 4 disjoint disks on∂D^3. Then, we glue them together. If we glue some 3 balls alone disks on their boundary, we get a handlebody. So P can be obtained by attaching m two handles on a handlebody of genus $n+1$. It is easy to see that m is the number of inner edges after gluing and n is the number of tetrahedron.

Now we double P its boundary to get a closed hyperbolic manifold DP. We have to control the rank of $\pi_1(DP)$. This is done in the following lemma.

Lemma 2.1. *Suppose P is obtained by attaching l-handles to a handlebody of genus k. Then $rk(\pi_1(DP)) \leq k + l$(DP is the double of P).*

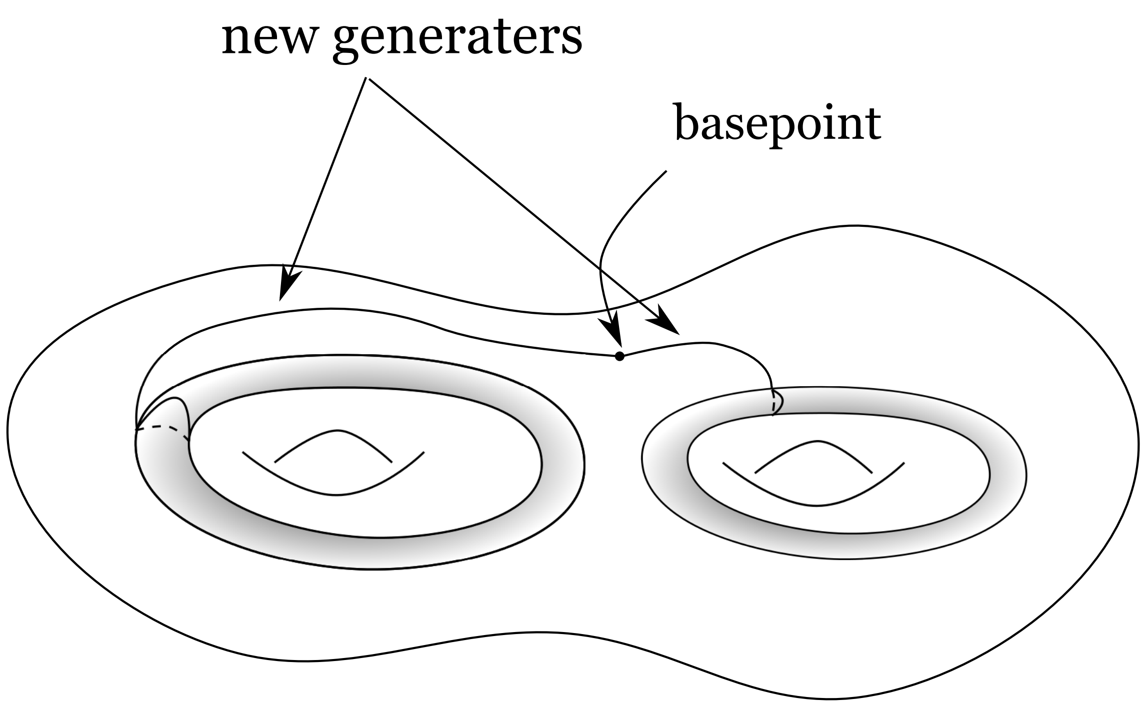

Figure 2

Now we can construct our examples. We start from n $(n > 3, 3 \nmid n)$copies of the tetrahedron indicated in Figure 3, where the edges are marked. We represent the faces by the edges around it. Each tetrahedron T_i has 4 faces $(1,3,2)_i$, $(4,5,3)_i$, $(2,6,4)_i$, $(5,1,6)_i$. Then we group the $4n$ faces into 2n pairs:

$$[(1,3,2)_i,(4,5,3)_{i+1}];\ [(2,6,4)_i,(5,1,6)_{i+1}],\ i = 1,2,....,n,\ \text{ and } n+1 \equiv 1.$$

The two faces in each pair are glued together respecting the order of vertex.

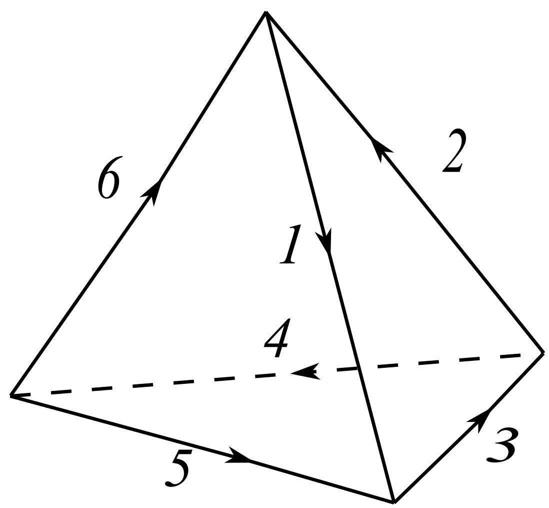

Figure 3

We can check that there are only one vertex and two edges after gluing. We denote the manifold by M. We double it to get DM..

Lemma 2.2. *In the construction above, $Fix(\phi) = Im(i_*(\pi_1(\partial M)))$.*

By Lemma 2.1, $\text{rk}(\pi_1 DM) \leq n+3$. Since ∂P has genus $n-1$, by Lemma 2.2 $Fix(\phi) = Im(i_*(\pi_1(\partial M))) \cong \pi_1(\partial M))$ has rank $2n-2$. For each $n > 2$, construct such pair (DM, ϕ), and denoted as (M_n, ϕ_n). Then

$$\frac{\text{rk}(Fix(\phi_n))}{\text{rk}(\pi_1(M_n))} \geq \frac{2n-2}{n+3} > 2-\epsilon, \text{ as } n \to \infty$$

for any $\epsilon > 0$. Hence we finished the proof of Theorem 1.4. ☑

The construction in Theorem 1.4 for closed hyperbolic 3-manifold can be modified to the case of hyperbolic 3-manifold with cusps.

3 Sketch Proof of Theorem 1.6

The most important tool is the following algebraic version of Mostow rigidity theorem. :

Theorem 3.1. *Let Γ_1 and Γ_2 be two cofinite volume klein groups. And $\phi : \Gamma_1 \to \Gamma_2$ is a isomorphism between them. Then there exist $\gamma \in Iso(\mathbb{H}^3)$ (γ may be orientation reversing) such that for any $\alpha \in \Gamma_1, \phi(\alpha) = \gamma\alpha\gamma^{-1}$.*

Now let's prove Theorem 1.6.

Proof. :

$$Fix(\phi) = \{\alpha \in G \mid \alpha\gamma = \gamma\alpha\}. \qquad (3.1)$$

Because $\gamma G \gamma^{-1} = G$, γ induces an isometry f of M, such that the following diagram commutes.

Case (1) γ is orientation preserving.

two nontrivial elements α, β in $Iso_+(\mathbb{H}^3)$ commute if and only if in the following cases:

(a) Both α and β are parabolic elements and they share the same fixed point in the infinite sphere S^∞;

(b) Both α and β are non-parabolic elements (elliptic or hyperbolic) and they share the same axis.

In both cases, if α, β, γ are all nontrivial, α commutes with β, β commutes with γ, then α commute with γ.

So $Fix(\phi)$ is a torsion free abelian group.

Case (2) γ is orientation reversing. Note $Fix(\phi) \subseteq Fix(\phi^2)$ and ϕ^2 is induced by an orientation preserving map. ☑

4 Sketch Proof of Theorem 1.5

Proposition 4.1. *Suppose M is a hyperbolic 3-manifold and S proper embedded surface in M. If there is an orientation reversing isometry f of order 2 on M fixing S pointwisly. Then*

$$rk(\pi_1(S)) < 2rk(\pi_1(M)).$$

Roughly speaking, it is a corollary of the following "Half die half alive lemma"

Lemma 4.2. *Suppose M is a compact orientable 3-manifold. Then*

$$dim\{image\ i^* : H^1(\partial M, \mathbb{Q}) \to H^1(M, \mathbb{Q})\} = \frac{dim H^1(\partial M, \mathbb{Q})}{2}$$

where i^* is induced by the inclusion $i : \partial M \to M$.

In order to prove the strict inequality, we need the following lemma and past to a finite covering space.

Lemma 4.3. *(D.Cooper,D.Long,A.Reid) Suppose M is a compact orientable 3-manifold which is not an I-bundle over surface, and S is an incompressible boundary component of M. Then there is a finite covering $p : \tilde{M} \to M$ such the $p^{-1}(S)$ contains more then one component.*

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