# Nuclei and exotic 4-manifolds

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## Abstract

(exotic := homeo but not diffeo)

$C^\infty$ structures of cpt 4-mfds with $\partial$ are not much investigated. (Relative SW invariants are difficult)

We introduce a new generalization of Gompf nuclei and give an application:

exotic smooth structures for a large class of 4-manifolds with boundary, regarding $C^0$ invariants and boundary 3-manifolds.

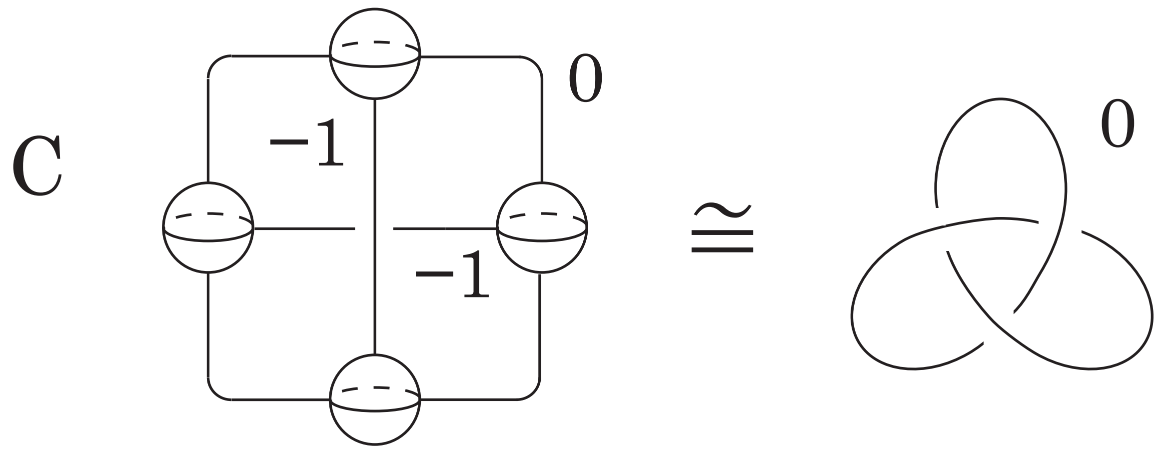
To detect smooth structures, we introduce a relative genus function.

# 1 Back ground

## Definition

(1) $C :=$ **cusp neighborhood**

i.e. $T^2$ fibration over $D^2$ with a single cusp fiber.




(2) A torus $T$ in a 4-mfd $X$ is $c$-**embedded**

$\Longleftrightarrow T$ is a regular fiber of $C \subset X$.

**Definition**

$X$ : a 4-mfd, $T$ : a $c$-embedded torus, $T \subset X$.

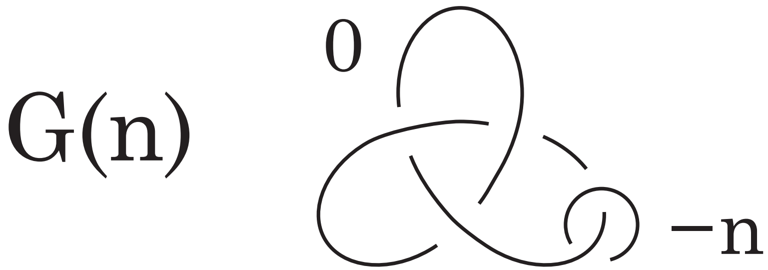
$$X_{(p)} := (X - \mathrm{int}\,\nu(T)) \cup_{\varphi_p} (T^2 \times D^2)$$

is called the **$p$-log transform** of $X$ along $T$.

**Definition**(Gompf '91, Ue '92)

$G(n) :=$ regular n.b.d of a cusp fiber and the section in the elliptic surface $E(n)$.

$G(n)$ $(n \geq 1)$ is called the **Gompf nucleus**.



## Remark

- $G(n)$ admits $\infty$ly many exotic smooth structures.
  (Gompf '91, Ue '92)
- Fuller ('99) generalized Gompf nuclei
  for higher genus fibrations.

## 2 A new generalization of nuclei

**Definition (Y)**

$N$ : a cpt. 4-mfd with $\partial$, $T$ : a torus, $T \subset N$.

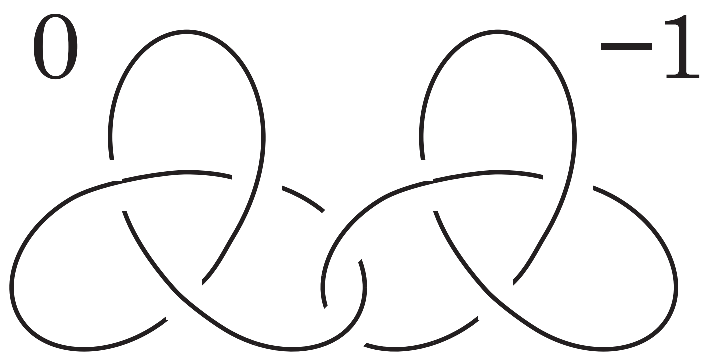
$(N, T)$ (or $N$ itself) is called a **nucleus** iff

- $T$ is $c$-embedded in $N$.
- $\pi_1(N) \cong 1$, $H_2(N) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $Q_N$ is unimodular.
- $\partial N$ is a <u>connected</u> homology 3-sphere.
- $\pi_1(N - \nu(T)) \cong \mathbb{Z}/d_T\mathbb{Z}$ $(d_T > 0)$.
  $d_T$ :=the **divisor** of $T$.
- $\pi_1(\partial\nu(T)) \to \pi_1(N - \nu(T))$ is surjective.

## Example of generalized nuclei

Put $X =$ cusp n.b.d. $\cup$ a 2-handle,
where the 2-handle is attached along a framed knot which links with the 0-framed trefoil geometrically once.

e.g.

X 


etc $\cdots$

## 3 Exotic structures on non-closed 4-manifolds

**Theorem A** (Y)

$X$ : arbitrary conn. ori. 4-mfd containing a nucleus.

( $X$ is closed,
or having (possibly disconnected) boundary,
or non-compact. )

Suppose $X \subset {}^{\exists}Z$ : closed oriented 4-manifold with $b_2^+ > 1$ and $SW_Z \neq 0$.

Then $X$ has $\infty$ly many exotic smooth structures.

- Construction: log transform (or FS knot surgery)

- The closed case follows from Fintushel-Stern's formula ('97) of SW invariants of log transforms.
- There are no known formulas of the relative SW invariants of log transform and knot surgery in this general condition on $X$ (particularly $\partial X$).
- When $\partial X$ is a certain $S^1$ bundle over surfaces, the relative SW formula of knot surgery was given under some conditions (Fintushel-Stern '05, Mark '11)

# 4 Application 1 of Thm A

## Exotic structures on cpt 4-mfds with $\partial$

**Definition**

$X$ is a (4-dim cpt ori.) **2-handlebody**

$\iff X =$ 0-handle $\cup$ 1-handles $\cup$ 2-handles

**Definition**

$X$ is a **compact Stein 4-manifold**

$\iff X$ is a 2-handlebody

s.t. its 2-handles are attached in a special way.

**Theorem B** (Y)

$N$ : a nucleus with a Stein structure.

$X$ : a 2-handlebody which contains the Stein handlebody $N$ as a sub-handlebody.

Then there exists a cpt conn. ori. 4-mfd $X_0$ s.t.

- $X_0$ has infinitely many exotic smooth structures.
- $\pi_1(X_0) \cong \pi_1(X), \quad H_*(X_0) \cong H_*(X),$
  $H_*(\partial X_0) \cong H_*(\partial X), \quad Q_{X_0} \cong Q_X.$
- $X_0$ (resp. $X$) can be embedded into $X$ (resp. $X_0$).

**Proof.** Application of corks and Thm A.

**Corollary B.1** (J. Park '07)

$G$: a finitely presented group

$\Longrightarrow {}^{\exists}X_0$: cpt conn. ori. 4-mfd with $\pi_1(X_0) \cong G$

s.t. $X_0$ has $\infty$ly many exotic smooth structures.

**Corollary B.2**

$Q : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z}$ : symmetric bilinear form $(k \geq 0)$

$R : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ : symmetric unimodular indefinite bilinear form.

$\Longrightarrow {}^{\exists}X_0$: cpt conn. ori. 4-mfd s.t.

- $X_0$ has infinitely many exotic smooth structures.
- $Q_{X_0} \cong Q \oplus R, \quad \pi_1(X_0) \cong 1.$

## 5 Application 2 of Thm A

## 3-manifolds bounding exotic 4-manifolds

**Question**

${}^{\forall}M$ : closed ori. 3-mfd,

${}^{\exists}X$ : cpt conn. ori. 4-mfd with $\partial X = M$

s.t. $X$ has $\infty$ly many smooth structures?

## Affirmative examples

- certain homology 3-spheres: Gompf '91, Ue '92,
- certain circle buncles over surfaces:
  Fintushel-Stern '05 and Mark '11

**Theorem C**

Suppose a closed ori. 3-mfd $M$ is either

- Stein fillable, or
- disjoint union of Stein fillable 3-mfds.

Then $M$ bounds a simp. conn. cpt. ori. 4-mfd with $\infty$ly many exotic smooth structures.

**Remark**

- Most 3-mfds are known to be Stein fillable.
- Ignoring $\pi_1$ of 4-mfds, we can also prove for some non-Stein fillable 3-mfds with both ori.

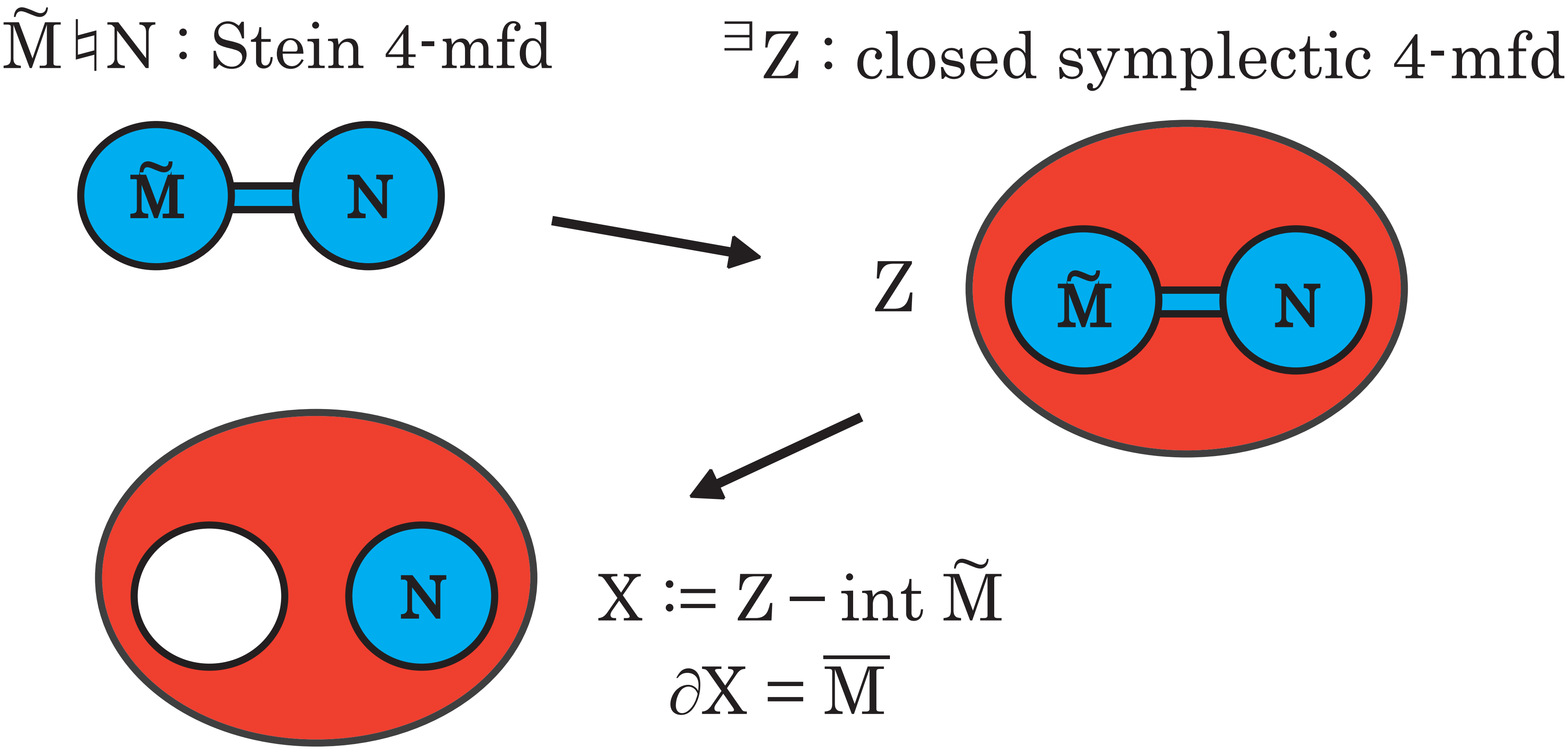
  e.g. $\Sigma(2,3,5)\#\overline{\Sigma(2,3,5)}$

# Proof of Thm C

$M$ : Stein fillable 3-mfd

$\widetilde{M}$ : cpt Stein 4-mfd with $\partial\widetilde{M} = M$

$N$ : nucleus which admits a Stein structure



# 6 Proof of Theorem A

**Theorem A** (Y)

$X$ : arbitrary conn. ori. 4-mfd containing a nucleus.

Suppose $X \subset {}^{\exists}Z$ : closed oriented 4-manifold with $b_2^+ > 1$ and $SW_Z \neq 0$.

Then $X$ has $\infty$ly many exotic smooth structures.

Candidates of the exotic smooth structures:

log transform $X_{(p)}$'s of $X$.

How can we detect smooth structures?

(When $X$ is not closed, relative SW is difficulut to compute)

We use genera of embedded surfaces in 4-manifolds.

**Fact**

$X$ : arbitrary connected oriented $C^\infty$ 4-manifold

$\Longrightarrow {}^\forall \alpha \in H_2(X;\mathbb{Z})$, ${}^\exists \Sigma_g \subset X$ s.t. $\alpha = [\Sigma_g]$.

( $\Sigma_g$ : a closed conn. ori. surface of genus $g$ )

The genus function $g_X : H_2(X) \to \mathbb{Z}$ is defined by

$$g_X(\alpha) = \min\{g \mid {}^{\exists}\Sigma_g \subset X \text{ s.t. } \alpha = [\Sigma_g]\}.$$

- $g_X$ is an invariant of smooth structures.
  (i.e. $g_X = g_Y \circ f_*$, where $f : X \to Y$ is a diffeo.)
- It is difficulut to see differences of genus functions due to automorphisms of $H_2(X)$.
- It is difficulut to determine (and evaluate) $g_X$.

## Relative genus function

Notation:

$$\mathbb{Z}^{\Lambda} = \{\mathbf{d} : \Lambda \to \mathbb{Z}\}, \quad (\mathbb{Z}_{\geq 0})^{\Lambda} = \{\mathbf{g} : \Lambda \to \mathbb{Z}_{\geq 0}\},$$

$$\mathrm{Sym}_{\Lambda}(\mathbb{Z})$$
$$= \{Q : \Lambda \times \Lambda \to \mathbb{Z} \mid {}^{\forall}\lambda_1, \lambda_2 \in \Lambda,\ Q(\lambda_1, \lambda_2) = Q(\lambda_2, \lambda_1)\}.$$

$X$ : arbitrary conn. ori. 4-manifold.

Assume $H_2(X)/\mathrm{Tor}$ is a free module, for simplicity.

$\Lambda$ : an index set bijective to a basis of $H_2(X)/\mathrm{Tor}$.

Fix an element $\lambda_1 \in \Lambda$.

**Definition (Y)**

For $Q \in \mathrm{Sym}_\Lambda(\mathbb{Z})$, $\mathbf{d} \in \mathbb{Z}^\Lambda$ $\mathbf{g} \in (\mathbb{Z}_{\geq 0})^{\Lambda - \{\lambda_1\}}$,

$G_X(Q, \mathbf{d}, \mathbf{g}) := \min\{g_X(\mathbf{d}_{\lambda_1}\mathbf{v}_{\lambda_1}) \mid \mathbf{v}$ satisfies $(*)\}$

$(*)$ $\mathbf{v} = \{\mathbf{v}_\lambda \mid \lambda \in \Lambda\}$ is a basis of $H_2(X)/\mathrm{Tor}$ satisfying

- The basis $\mathbf{v}$ represents $Q$ (i.e. $\mathbf{v}_\lambda \cdot \mathbf{v}_\mu = Q(\lambda, \mu)$);
- ${}^\forall\lambda \in \Lambda - \{\lambda_1\}$, $g_X(\mathbf{d}_\lambda \mathbf{v}_\lambda) \leq \mathbf{g}_\lambda$.

When ${}^\nexists$ basis $\mathbf{v}$ satisfying $(*)$, $G_X(Q, \mathbf{d}, \mathbf{g}) := \infty$.

We call $G_X : \mathrm{Sym}_\Lambda(\mathbb{Z}) \times \mathbb{Z}^\Lambda \times (\mathbb{Z}_{\geq 0})^{\Lambda - \lambda_1} \to \mathbb{Z} \cup \{\infty\}$ the **relative genus function** of $X$.

$G_X$ is an invariant of smooth structures.

i.e. if $X$ and $Y$ are diffeo, then $G_X = G_Y$.

(No need to consider automorphisms of $H_2(X)$)

## Proof of Thm A

For a certain $(Q, \mathbf{d}, \mathbf{g})$,
Using the adjunction inequality, we obtain:

$$G_{X_{(p)}}(Q, \mathbf{d}, \mathbf{g}) < G_{X_{(p')}}(Q, \mathbf{d}, \mathbf{g}) \; ({}^\forall p' \gg p).$$

Therefore, if $p_1 \ll p_2 \ll \cdots \ll p_n \ll \cdots$,

then $X_{(p_i)}$ $(i \geq 1)$ are mutually non-diffeo.