$\bar{\mu}$ invariant of nanophrases

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Introduction

Knot theory

Link

$\Longrightarrow$

Topology of words

Nanophrase

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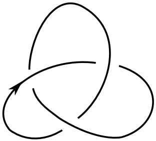

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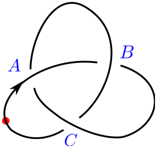

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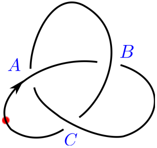

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ABCABC

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Link / Link homotopy

Milnor's $\bar{\mu}$ invariant

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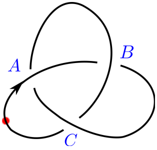
Nanophrase / M-homotopy

extended $\bar{\mu}$ invariant



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Link homotopy

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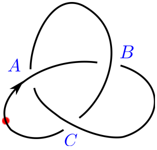

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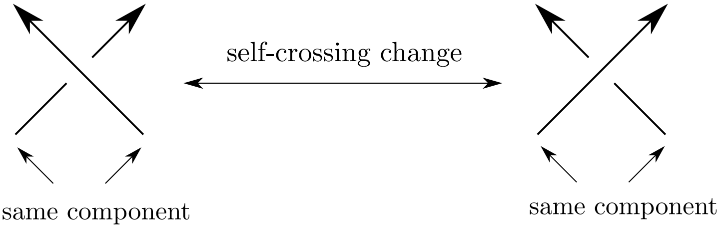
Link homotopy

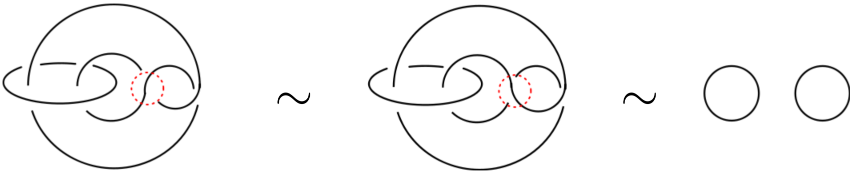


Definition 1 (Milnor)

Link homotopy :
an equivalence relation of links generated by ambient isotopy and self-crossing change.

Example of link homotopy

Example (Whitehead link)



$\bar{\mu}$ invariant ([Milnor])

G : a group
G_q : a qth lower central group of G

Theorem 2 (Chen, Milnor)

L : an ordered oriented n-component link

$$\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))_q \cong \langle x_1, x_2, \ldots, x_n \mid [x_1, l_1], \ldots, [x_n, l_n], A_q \rangle$$

x_i : a meridian of ith component
l_i : a longitude of ith component
$A = \langle x_1, x_2, \ldots, x_n \rangle$: a free group

$\varphi : \langle x_1, x_2, \ldots, x_n \rangle \to \mathbb{Z}[[x_1, x_2, \ldots, x_n]]$: Magnus expansion

$$\begin{aligned} \varphi(x_i) &= 1 + x_i \\ \varphi(x_i^{-1}) &= 1 - x_i + {x_i}^2 - {x_i}^3 + \cdots \end{aligned}$$

$$\varphi(l_i) = 1 + \sum \mu(c_1, c_2, \ldots, c_u, i)\, x_{c_1} x_{c_2} \ldots x_{c_u}$$

$c_1, c_2, \ldots, c_u, i$ is a sequence of integers between 1 and n

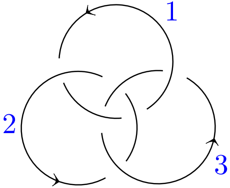
$$\bar{\mu}(c_1, c_2, \ldots, c_u, i) := \mu(c_1, c_2, \ldots, c_u, i) \mod \Delta(c_1, c_2, \ldots, c_u, i)$$

Δ is the greatest common divisor of some set of μ

Theorem 3 (Milnor)

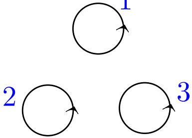
Let L be an ordered oriented link. Let $c_1, c_2, \ldots, c_u, i$ be a sequence of integers between 1 and n such that the indices $c_1, c_2, \ldots, c_u, i$ are pairwise distinct. Then $\bar{\mu}(c_1, c_2, \ldots, c_u, i)$ is a link homotopy invariant.

Example (Borromean ring)



$\bar{\mu}(3,2,1) = 1$

$\not\sim$



$\bar{\mu}(3,2,1) = 0$

Nanophrase and homotopy

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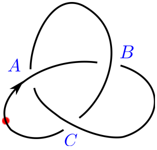

$\Longrightarrow$ ABCABC

Gauss phrase

alphabet : a finite set $(\mathcal{A} = \{A, B, C\})$

Gauss phrase on $\mathcal{A}$:
a finite sequence of words on $\mathcal{A}$ such that each letter appears exactly twice

Example:

$ABA|B$ $(\mathcal{A} = \{A, B\})$,

$ABA|B$ $(\mathcal{A} = \{A, B, C\})$ not Gauss phrase on $\mathcal{A}$

Nanophrase

$\mathcal{A}$: an alphabet
α : a finite set
$|\cdot| : \mathcal{A} \longrightarrow \alpha$

n-component nanophrase $(\mathcal{A}, p)$ over α :

- $\mathcal{A}$: alphabet with a map $|\cdot|$
- p : Gauss phrase on $\mathcal{A}$

Example :

$\mathcal{A} = \{A, B, C\}, \ \ \alpha = \{a, b\}$
$|A| = |B| = a$, $|C| = b$
$p = ABA|BCC$

Homotopy

Definition 4 (Turaev)

homotopy :
an equivalence relation of nanophrases over α generated by isomorphism, H1, H2, H3 moves and shift move

isomorphism $\longleftrightarrow$ changing letters in nanophrase
H1, H2, H3 moves (with τ, S) $\longleftrightarrow$ Reidemeister moves
shift move (with ν) $\longleftrightarrow$ changing base points

Remark
Homotopy depends on a choice of α, τ, S and ν (homotopy data).

$\alpha = \{a_+, a_-, b_+, b_-\}$
τ : an involution on α $\quad (a_+ \mapsto b_-,\ a_- \mapsto b_+)$

$$S = \left\{ \begin{array}{l} (a_+, a_+, a_+), (a_+, a_+, a_-), (a_+, a_-, a_-), \\ (a_-, a_-, a_-), (a_-, a_-, a_+), (a_-, a_+, a_+), \\ (b_+, b_+, b_+), (b_+, b_+, b_-), (b_+, b_-, b_-), \\ (b_-, b_-, b_-), (b_-, b_-, b_+), (b_-, b_+, b_+) \end{array} \right\}$$

ν : an involution on α $\quad (a_+ \mapsto b_+,\ a_- \mapsto b_-)$

Theorem 5 (Turaev)

There exists a bijection:

$$\left\{ \begin{array}{c} \textit{ordered oriented} \\ \textit{virtual links} \end{array} \right\} \longleftrightarrow \{ \textit{ nanophrases over } \alpha\} / \textit{ homotopy}$$

M-homotopy

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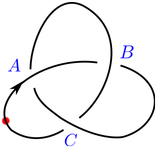

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M-homotopy

$(\mathcal{A}, p)$: a nanophrase over α
σ : an involution on α

- self-crossing move :
  - not change Gauss phrase
  - if $...|...A...A...|...$ then change $|A|$ to $\sigma(|A|)$

Definition 6 (K.)

M-homotopy :
an equivalence relation of nanophrases over α generated by homotopy and self-crossing move

$\bar{\mu}$ of nanophrase

Main theorem (K.)

Let $(\mathcal{A}, p)$ be a nanophrase. Let $c_1, c_2, \ldots, c_u, i$ be a sequence of integers between 1 and n such that the indices $c_1, c_2, \ldots, c_u, i$ are pairwise distinct. Then $\bar{\mu}(c_1, c_2, \ldots, c_u, i)$ is an invariant under M-homotopy of nanophrases with respect to (α, τ, S), ν and σ.

$\alpha = \{a_+, a_-, b_+, b_-\}$

τ : an involution on α $(a_+ \mapsto b_-,\ a_- \mapsto b_+)$

$$S = \left\{ \begin{array}{l} (a_+, a_+, a_+), (a_+, a_+, a_-), (a_+, a_-, a_-), \\ (a_-, a_-, a_-), (a_-, a_-, a_+), (a_-, a_+, a_+), \\ (b_+, b_+, b_+), (b_+, b_+, b_-), (b_+, b_-, b_-), \\ (b_-, b_-, b_-), (b_-, b_-, b_+), (b_-, b_+, b_+) \end{array} \right\}$$

ν : an involution on α $(a_+ \mapsto b_+,\ a_- \mapsto b_-)$

σ : an involution on α $(a_+ \mapsto a_-,\ b_+ \mapsto b_-)$

Corollary 7 (K.)

Let L be an ordered oriented virtual link. Let $c_1, c_2, \ldots, c_u, i$ be a sequence of integers between 1 and n such that the indices $c_1, c_2, \ldots, c_u, i$ are pairwise distinct. Then $\bar{\mu}(c_1, c_2, \ldots, c_u, i)$ is a link homotopy invariant.

Remark

Dye, H. A. and Kauffman, L. H.
Kravchenko, O. and Polyak, M.

Thank you!