

A property of normalized arrow polynomials of checkerboard colorable virtual links

Takanori IMABEPPU

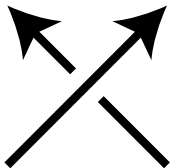
Hirosima University, D1

Definition

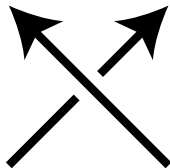
virtual link diagram

Def

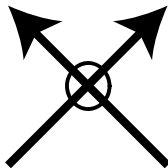
A **virtual link diagram** is a link diagram which may have virtual crossings.



positive



negative



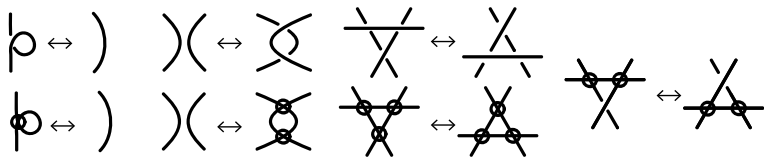
virtual

Definition

virtual link

Def

A **virtual link** is the equivalence class of virtual link diagram under the generalized Reidemeister moves.

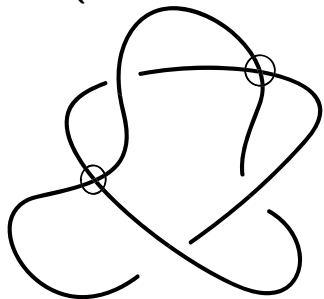


Generalized Reidmeister moves

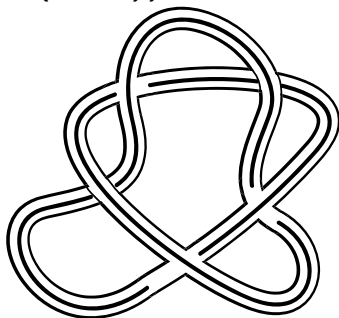
Definition

abstract link diagram

Def(abstract link diagram(ALD))



virtual link diagram D



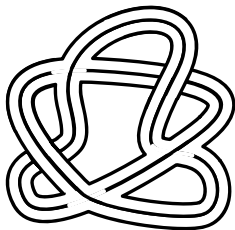
ALD associated with D

Definition

checkerboard coloring

Def

A **checkerboard coloring** of ALD is a coloring of regions by black and white, such that adjacent regions have different colorings.



Definition

checkerboard coloring

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Definition

checkerboard colorable

Def

D : a virtual link diagram.

D is **checkerboard colorable**

\iff an ALD associated with D is checkerboard colorable.

Definition

checkerboard colorable

Def

D : a virtual link diagram.

D is **checkerboard colorable**

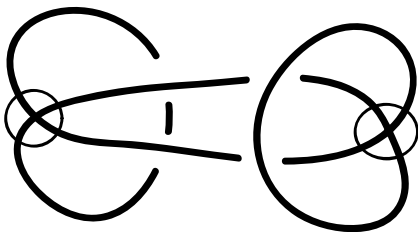
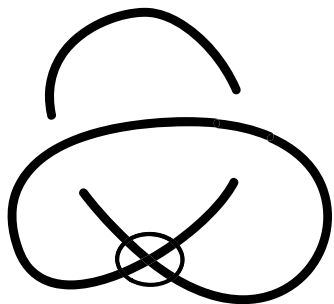
\iff an ALD associated with D is checkerboard colorable.

Fact

Any classical link diagram is checkerboard colorable.

Example

EX



Definition

checkerboard colorable

Def

L : a virtual link.

L is **checkerboard colorable**

$\iff L$ has a checkerboard colorable virtual link diagram.

We show that a certain virtual link is not checkerboard colorable by using invariants for virtual links.

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In this talk, a virtual link is always oriented.

f-polynomial

Fact[Kauffman '99]f-polynomial

$$f: \{\text{virtual link}\} \longrightarrow Z[A, A^{-1}]$$

f-polynomial

Fact[Kauffman '99]f-polynomial

$$f: \{\text{virtual link}\} \longrightarrow Z[A, A^{-1}]$$

Thm[N.Kamada '02]

L : a virtual link with n -components.

$f(L)$: a f-polynomial of L .

$\text{EXP}(f(L))$: the set of integers appearing as exponents of $f(L)$.

$$\text{ex) } f(L) = 2A^6 + A^4 - A^{-2}$$

$$\text{EXP}(f(L)) = \{6, 4, -2\}$$

f-polynomial

Fact[Kauffman '99] f-polynomial

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Thm[N.Kamada '02]

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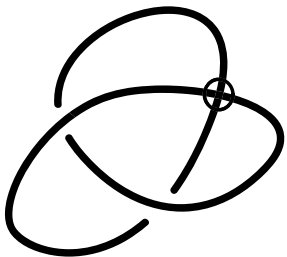
L is checkerboard colorable

$$\implies \text{EXP}(f(L)) \subset \begin{cases} 4Z & n \text{ is odd} \\ 4Z + 2 & n \text{ is even} \end{cases}$$

Example

f-polynomial

EX



$$f(L) = A^{-4} + A^{-6} - A^{-10}$$
$$\text{EXP}(f(L)) = \{-4, -6, -10\}$$

Definition

normalized arrow polynomial

L : a virtual link with n -components.

D : a virtual link diagram of L .

A state S of D is obtained by choosing A-splice or B-splice for each crossing in D .

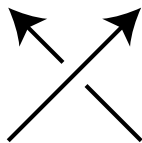
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positive

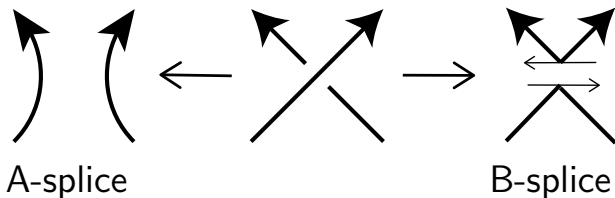
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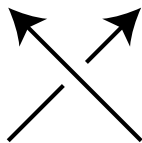
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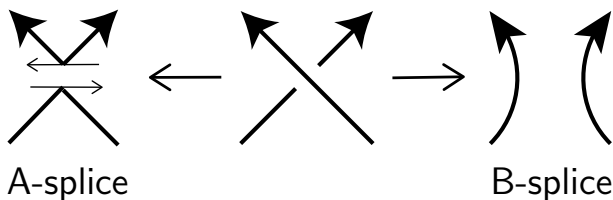
Definition

normalized arrow polynomial

L : a virtual link with n -components.

D : a virtual link diagram of L .

A state S of D is obtained by choosing A-splice or B-splice for each crossing in D .



Definition

normalized arrow polynomial

s : a state of D .

$\sigma(s) = \#\{\text{A-splice}\} - \#\{\text{B-splice}\}$.

$l(s)$ is the number of components of s .

An arrow polynomial of D is defined by

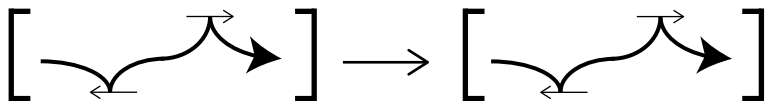
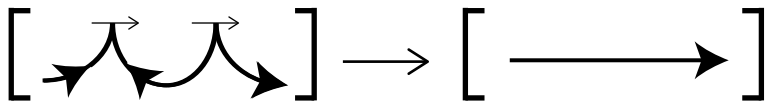
$$\mathcal{A}(D) := \sum_s A^{\sigma(s)} (-A^2 - A^{-2})^{l(s)-1} \mathcal{K}(s).$$

$\mathcal{K}(s)$ is a product of extra variables $K_1, K_2 \cdots$ associated with the state s .

Definition

normalized arrow polynomial

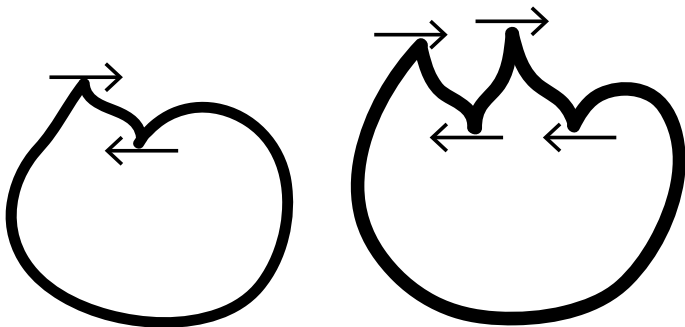
State Reduction



Definition

normalized arrow polynomial

After State Reducation, s consists of some loops with even alternating arrows.



Definition

normalized arrow polynomial

$\{l_1, \dots, l_p\}$: the set of the loops of s .
 l_i has $2m_i$ alterneing arrows.

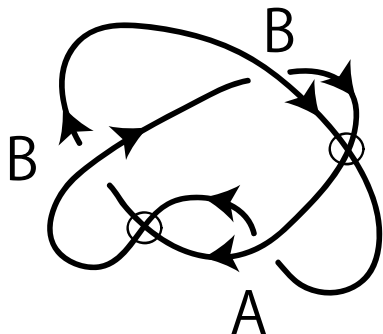
$$\mathcal{K}(s) := K_{m_1} K_{m_2} \cdots K_{m_p}.$$

If $m_i = 0$, then $K_{m_i} := 1$.

Definition

normalized arrow polynomial

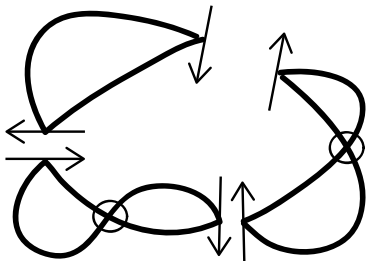
EX



Definition

normalized arrow polynomial

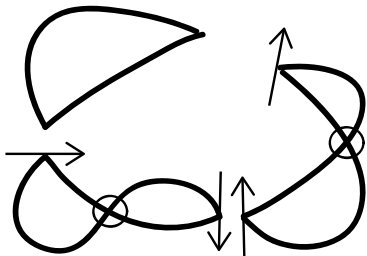
EX



Definition

normalized arrow polynomial

EX



$$\mathcal{K}(s) = K_0 K_1 K_1 = 1 K_1 K_1 = K_1^2.$$

Definition

normalized arrow polynomial

Thm[Kauffman '10]

$$\mathcal{A}(D) := \sum_s A^{\sigma(s)} (-A^2 - A^{-2})^{l(s)-1} \mathcal{K}(s).$$

$\omega(D)$: the writhe of D .

A normalized arrow polynomial of D is defined by

$$\mathcal{W}(D) := (-A^{-3})^{\omega(D)} \mathcal{A}(D).$$

$\mathcal{W}(D)$ is an invariant for virtual links.

normalized arrow polynomial

I introduce two methods of detecting checkerboard colorability of virtual links by using a normalized arrow polynomial.

normalized arrow polynomial

Thm1

L : a virtual link with n -components.

$\mathcal{W}(L)$: a normalized arrow polynomial of L .

L is checkerboard colorable

$$\implies \text{EXP}(\mathcal{W}(L)) \subset \begin{cases} 4Z & n \text{ is odd} \\ 4Z + 2 & n \text{ is even} \end{cases}$$

normalized arrow polynomial

$K_{n_1}^{m_1} K_{n_2}^{m_2} \dots K_{n_t}^{m_t} \neq 1$ is good.

$\iff \exists a_{ij}$: sequence s.t.

$$\sum_{i=1}^t \left(n_i \sum_{j=1}^{m_i} a_{ij} \right) = 0, \quad a_{ij} = \pm 1.$$

1 is good.

normalized arrow polynomial

EX $K_1^2 K_2$ is good.

$$) 1 + 1 - 2 = 0.$$

normalized arrow polynomial

EX

$K_1^2 K_2$ is good.

$$) 1 + 1 - 2 = 0.$$

$K_1^2 K_2 K_3$ is not good.

normalized arrow polynomial

EX

$K_1^2 K_2$ is good.

$$) 1 + 1 - 2 = 0.$$

$K_1^2 K_2 K_3$ is not good.

K_1^2 is good.

K_2 is not good.

normalized arrow polynomial

Thm2

L : a virtual link.

$\mathcal{W}(L)$: a normalized arrow polynomial of L .

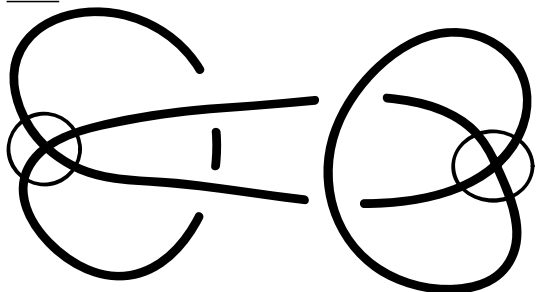
If L is checkerboard colorable, then any term of $\mathcal{W}(L)$ is good.

normalized arrow polynomial

Thm

The method of detecting checkerboard colorability of virtual links by using Thm2 is stronger than that by using Thm1.

normalized arrow polynomial

EX

$$\mathcal{W}(L) = 1 + A^4 + A^{-4} - (2 + A^4 + A^{-4})K_1^2 + 2K_2.$$