# Bounds for fixed points on Seifert manifolds

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# Abstract

In this talk, we consider homeomorphisms of compact connected orientable Seifert manifolds, and give some bounds involving the rank and the index of fixed point classes. One consequence is a index bound for fixed point classes. We rely on the classification of 2-orbifolds homeomorphisms and the similar bound on surfaces.

# Fixed point class

Let $X$ be a connected compact polyhedron, and $f : X \to X$ a self-map.

## Definition

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path $c$ (called a Nielsen path) from $x$ to $x'$ such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x).$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

# Rank of a fixed point class

## Definition

For a fixed point $x \in \mathbf{F}$, let

$$\mathrm{Stab}(f, x) := \{\gamma \in \pi_1(X, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x),$$

where $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ is the induced endomorphism. It is independent of the choice of $x \in F$, up to isomorphism. For a fixed point class $\mathbf{F}$ of $f$ , define the rank to be

$$\mathrm{rank}(\mathbf{F}) := \mathrm{rank}(f, x) := \mathrm{rank}\mathrm{Stab}(f, x), \quad \forall x \in \mathbf{F}.$$

For an empty fixed point class $\mathbf{F}$, we can make it nonempty by deforming $f$.

# Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of $f_0$ to the fixed point classes of $f_1$.
**Remark.** A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

**Theorem (Homotopy invariance)**

*Under the correspondence via a homotopy $H$,*

$$\text{ind}(f_0, \mathbf{F}_0) = \text{ind}(f_1, \mathbf{F}_1), \quad \text{rank}(f_0, \mathbf{F}_0) = \text{rank}(f_1, \mathbf{F}_1).$$

# Characteristic of a fixed point class

From now on, unless otherwise stated, we always assume $X$ to be a graph, a surface or a Seifert manifold, and $f : X \to X$ is a selfmap. For convenience, we define another term.

**Definition**

The characteristic of a fixed point class $\mathbf{F}$ is defined as

$$\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rank}(\mathbf{F}).$$

The only exception is when $X$ is a closed surface, $f$ is homotopic to the identity map, and $\mathbf{F}$ corresponds (via the homotopy) to the whole of $X$, in this case

$$\mathrm{chr}(\mathbf{F}) := \chi(X) = 2 - \mathrm{rank}(\mathbf{F}).$$

# Bounds for Seifert manifolds

Our main result is

**Theorem (Main Theorem)**

*Suppose $M$ is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$ and $k$ ($k \geq 0$) singular fibers, and $f : M \to M$ is a homeomorphism. Then*
(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* $f$*;*
(B) $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq \mathcal{B}$*, where*

$$\mathcal{B} = \begin{cases} 4\big(3 - \mathrm{rank}\pi_1(M)\big) & \partial M = \Phi, k = 0 \\ 4\big(1 - \mathrm{rank}\pi_1(M)\big) & M = S^2(\alpha_1, \dots, \alpha_k) \\ 4\big(2 - \mathrm{rank}\pi_1(M)\big) & \textit{others} \end{cases},$$

*where $k \geq 4$ is even, $\alpha_i$ is odd for some $i$, and $\alpha_j = 2$ for $j \neq i$ in the second case, and the sum is taken over all essential fixed point classes* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

# Bounds on index

As a corollary, we have

**Theorem (Bounds on index)**

*Suppose $M$ is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$ and $k$ $(k \geq 0)$ singular fibers, and $f : M \to M$ is a homeomorphism. Then the index of each fixed point class of $f$ has bounds. More precisely,*

$$\sum_{\mathrm{ind}(\mathbf{F})+1<0} \{\mathrm{ind}(\mathbf{F}) + 1\} \geq \mathcal{B}.$$

This bound is similar to the one on graphs and surfaces. For $f$ is orientation preserving, B. Jiang and S. Wang proved that the index of each essential fixed point class of $f$ is $\pm 1$.

# Bounds on rank

Moreover, we can get an immature bounds on the rank of fixed subgroups.

## Proposition (Bounds on rank)

*Suppose* $f : M \to M$ *is a homeomorphism of a compact connected orientable Seifert manifold with hyperbolic orbifold* $X(M)$*. Let* $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ *is the induced automorphism and* $\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(X, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x)$*, where* $x$ *is in a essential fixed point class. Then*

$$\mathrm{rank}\mathrm{Fix}(f_\pi) \leq 2\mathrm{rank}\pi_1(M).$$

**Remark.** If $M$ is a Seifert manifold and $\phi : \pi_1(M) \to \pi_1(M)$ is an automorphism, the Proposition dose not holds, and S. Wang give an counter example such that the rank of the fixed subgroup is infinite,

# An example

The example bellow shows that the bound in the Main Theorem is sharp.

**Example.**
Let $M = F_2 \times S^1$, where $F_2$ is an orientable closed surface of genus 2 and $S^1 = \{e^{i\theta} | \theta \in (-\pi, \pi]\}$ is a circle. Give coordinates of $F_2$ as following.

$$F_2 = (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int} D_1) \bigcup_g (\partial D_2 \times [0,1]) \bigcup_h (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int} D_2),$$

where $\varphi, \psi, \alpha, \beta \in (-\pi, \pi]$, and $D_1 = \{e^{i\varphi} \times e^{i\psi} | \varphi, \psi \in [-\pi/4, \pi/4]\}$, $D_2 = \{e^{i\alpha} \times e^{i\beta} | \alpha, \beta \in [-\pi/4, \pi/4]\}$ are two disks, and

$$g : \partial D_1 \to \partial D_2 \times [0,1], \qquad g(e^{i\varphi}, e^{i\psi}) = (e^{i\alpha}, e^{i\beta}, 0),$$

$$h : \partial D_2 \times \{1\} \to \partial D_2, \qquad h(e^{i\alpha}, e^{i\beta}, 1) = (e^{i\alpha}, e^{i\beta}).$$

# An example

Let $f = f_L \bigcup f_M \bigcup f_R : M \to M$, where

$$f_L : (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int} D_1) \times S^1 \to (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int} D_1) \times S^1,$$

$$f_L(e^{i\varphi}, e^{i\psi}, e^{i\theta}) = (e^{i\varphi}, e^{i\psi}, e^{-i\theta}),$$

$$f_R : (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int} D_2) \times S^1 \to (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int} D_2) \times S^1,$$

$$f_R(e^{i\alpha}, e^{i\beta}, e^{i\theta}) = (e^{i\alpha}, e^{i\beta}, e^{i(\alpha-\theta)}),$$

and

$$f_M : (\partial D_2 \times [0,1]) \times S^1 \to (\partial D_2 \times [0,1]) \times S^1,$$

$$f_M(e^{i\alpha}, e^{i\beta}, t, e^{i\theta}) = (e^{i\alpha}, e^{i\beta}, t, e^{i(t\alpha-\theta)}).$$

So

$$\mathrm{Fix}(f) = \mathrm{Fix}(f_L) \bigcup \mathrm{Fix}(f_M) \bigcup \mathrm{Fix}(f_R) \cong F_3,$$

an orientable closed surface of genus 3.

# An example

Hence $f$ has unique essential fixed point class $\mathbf{F} \cong F_3$,

$$\operatorname{ind}(\mathbf{F}) = \operatorname{chr}(\mathbf{F}) = -4,$$

$$\operatorname{rank}\operatorname{Fix}(f_\pi) = 6 > 5 = \operatorname{rank}\pi_1(M),$$

and

$$\sum_{\operatorname{ind}(\mathbf{F})+\operatorname{chr}(\mathbf{F})<0} \{\operatorname{ind}(\mathbf{F}) + \operatorname{chr}(\mathbf{F})\} = 4\big(3 - \operatorname{rank}\pi_1(M)\big).$$

# Outline of the proof of Main Theorem

We first consider the circle bundle over an orientable surface.

**Lemma**

*Suppose* $q : M \to F$ *is an orientable circle bundle over an orientable compact surface* $F$ *with* $\chi(F) < 0$. *If* $f : M \to M$ *is a fiber preserving homeomorphism that reverses the fiber orientation, then*

(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* $f$*;*

(B)

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(F),$$

*where the sum is taken over all essential fixed point class* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

# Outline of the proof of Main Theorem

**Proof.**

Via a fiber preserving homotopy, we can suppose the induced map $f' : F \to F$ is in "standard form". Let $C'$ be a component of $\mathrm{Fix}(f')$, and let $C = q^{-1}(C') \cap \mathrm{Fix}(f)$. Then we have the fact that $C'$ is doubly covered by $C$ and the fixed point classes of $f$ are connected. The conclusions follow easily form the bounds on hyperbolic surfaces in [JWZ]:

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(F).$$

□

# Outline of the proof of Main Theorem

Now we consider Seifert manifolds.

**Proposition**

*Suppose $M$ is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$, Then*
(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* $f$*;*
(B)

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(X(M))$$

*where the sum is taken over all essential fixed point class* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

# Outline of the proof of Main Theorem

**Sketch of proof:**
Since $\mathrm{ind}(\mathbf{F})$ and $\mathrm{chr}(\mathbf{F})$ are homotopy invariants, up to a isotopy and a fiber preserving isotopy, we may assume $f$ is fiber preserving with respect to this fibration and reverses the orientation of any fiber which contains a essential fixed point of $f$.

Since $X(M)$ is hyperboloc, there is a finite covering $q : X(\widetilde{M}) \to X(M)$ of orbifold such that $X(\widetilde{M})$ is an orientable surface. The pull-back of the Seifert fibration $p : M \to X(M)$ via $q$ gives a covering manifold $q' : \widetilde{M} \to M$ with fibration $p' : \widetilde{M} \to X(\widetilde{M})$. Since $X(\widetilde{M})$ is an orientable surface, the fibration $p'$ is an orientable circle bundle over the orientable surface $X(\widetilde{M})$. After passing to further finite covering if necessary, we may assume that $q'$ is characteristic.

# Outline of the proof of Main Theorem

Let $\tilde{f} : \widetilde{M} \to \widetilde{M}$ be a lifting of $f$ which fixes a point $A'$ with $q'(A') = A$. Since $A$ is a fixed point of $f$, by the above assumption $f$ reverses the orientation of the fiber containing $A$. As a lifting of $f$, the map $\tilde{f}$ reverses the orientation of the fiber containing $A'$, and hence the orientation of all fibers because the fibers of $\widetilde{M}$ can be coherently oriented. Since the orbifold homeomorphism $\tilde{f}' : X(\widetilde{M}) \to X(\widetilde{M})$ is a lifting of the standard map $f'$, it is also standard. So $\tilde{f}$ and $f$ both have the FR-property. Hence, a essential fixed point class of $f$ (resp. $\tilde{f}$) is a connected component of $\mathrm{Fix}(f)$ (resp. $\mathrm{Fix}(\tilde{f})$).
Let $\mathcal{F} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_d\}$ be all the liftings of $f$, and $\mathcal{F}' = \{\tilde{f}'_1, \tilde{f}'_2, \ldots, \tilde{f}'_d\}$ be all the liftings of $f'$, where $\tilde{f}'_i$ is the induced homeomorphism on orbifold of $\tilde{f}_i$. For any fixed point class $\mathbf{F}$ of $f$ and any fixed point class $\widetilde{\mathbf{F}}_i$ of $\tilde{f} \in \mathcal{F}$ which is a $k_i$-fold cover of $\mathbf{F}$, we have

$$\mathrm{chr}(\tilde{f}, \widetilde{\mathbf{F}}_i) = k_i \mathrm{chr}(f, \mathbf{F}), \mathrm{ind}(\tilde{f}, \widetilde{\mathbf{F}}_i) = k_i \mathrm{ind}(f, \mathbf{F}).$$

So equality (A) follows from $\mathrm{ind}(f, \mathbf{F}) \le \mathrm{chr}(f, \mathbf{F})$ on circle bundle.

# Outline of the proof of Main Theorem

Now we consider equality (B). By the discussion above, we have

$$\begin{aligned}
&\sum_{\tilde{f}\in\mathcal{F}} \sum_{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0} \{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)\} \\
&= \sum k_i \sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})\} \\
&= d \sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})\}
\end{aligned}$$

where the sum is taken over all fixed classes $\widetilde{\mathbf{F}}_i$ and all lifting $\tilde{f}$ of $f$ with $\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0$, and

$$\begin{aligned}
&\sum_{\tilde{f}\in\mathcal{F}} \sum_{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0} \{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)\} \\
&\geq 2 \sum_{\tilde{f}'\in\mathcal{F}'} \sum_{\mathrm{ind}(\tilde{f}',\widetilde{\mathbf{F}}'_i)+\mathrm{chr}(\tilde{f}',\widetilde{\mathbf{F}}'_i)<0} \{\mathrm{ind}(\tilde{f}',\widetilde{\mathbf{F}}'_i)+\mathrm{chr}(\tilde{f}',\widetilde{\mathbf{F}}'_i)\} \\
&\geq 4d\chi(X(M)).
\end{aligned}$$

# Outline of the proof of Main Theorem

Hence

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(X(M)),$$

where the sum runs over all essential fixed point class $\mathbf{F}$ with $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$. So equality (B) holds.

# Outline of the proof of Main Theorem

The proof is finished by estimating rank$\pi_1(M)$ and $X(M)$ from the following theorem.

**Theorem (Zieschang, Peczynski, Rosenberger)**

*Let $G$ be a Fuchsian group with orbifold $\mathcal{O} = F(\alpha_1, \ldots, \alpha_k)$, where the closed surface $F$ denotes the underlying space of $\mathcal{O}$ and $\alpha_i > 1$ denotes the order of the cone points. Then the following hold.*

(1) *If* $k = 0$ *then* $\mathrm{rank}(G) = -\chi(F) + 2$;

(2) *If* $\chi(F) = 2$, *that is,* $F = S^2$, $k \geq 4$ *is even,* $\alpha_i$ *is odd for some* $i$, *and* $\alpha_j = 2$ *for* $j \neq i$, *then* $\mathrm{rank}(G) = -\chi(F) + k = k - 2$;

(3) *In all other cases* $\mathrm{rank}(G) = -\chi(F) + k + 1$.

Thank you